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The many colors of algebra: The impact of equity focused teaching upon student learning and engagement

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ABSTRACT

The number of students who leave U.S. schools mathematically underprepared has prompted widespread concern. Low achieving students, many of whom have been turned off mathematics, are often placed in low tracks and given remedial, skills-oriented work. This study examines a different approach wherein heterogeneous groups of students were given responsibility and agency and asked to engage in a range of mathematical practices collaboratively. The teaching intervention, which was introduced in the first paper, took place as part of a summer class on algebra, and it gave students the opportunity to participate with mathematics in changed ways. This paper will report evidence that the vast majority responded with increased engagement, achievement, and enjoyment. The students chose collaboration and agency as critical to their improved relationships with mathematics.

1. Introduction

The number of students in the United States who dislike or fear mathematics and leave school mathematically underprepared has prompted widespread concern (Boaler, 2015; Glenn, 2000). Many students – even those who are successful – develop negative ideas about math and see the subject as something that is ultimately uninteresting and quite separate from their lives (Boaler & Greeno, 2000; Madison & Hart, 1990; Seymour & Hewitt, 1997). Additionally, mathematics has a wider "gap" across socio-economic and racial lines than any other academic subject (RAND, 2002; Secada, Fennema, & Adajian, 1995; Tate, 1997). Persistent failure and disinterest in mathematics is of particular concern given the growing importance of mathematical reasoning and 'quantitative literacy' (Boaler, 2013b; Steen, 1997) to people's lives and work, and an increasing knowledge of the ways that unsuccessful mathematics experiences can impact students well beyond the classroom (Boaler, 2005; Moses & Cobb, 2001; Thompson, 1995).

Despite the development of mathematics education as a growing field of research in the last few decades, and the identification of features of learning environments that bring about mathematical interest and high achievement, traditional teaching of mathematics endures (Hibert & Stigler, 2000; Rosen, 2001). Indeed there is a large body of research in mathematics education that shows the ways students can learn mathematics most effectively, with evidence of increased engagement and achievement success, that is not taken up in classrooms. There are many reasons for the gap between what we know works and what is used in classrooms, one of them being the inaccessibility of research knowledge that is largely

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contained in journals teachers do not read. New technologies giving teachers access to research knowledge are helping to cross the research-practice divide, especially when they translate research knowledge into useable resources and ideas for teaching (see, for example, www.youcubed.org). Positive characteristics of mathematics classrooms include reasoning about applied problems, discussing mathematical ideas, and actively engaging in mathematical learning (e.g., Boaler, 2002a; Boaler & Staples, 2008; Kieran, 1994; Malloy, 2009). As Gutstein (2003) demonstrates, for example, inviting students to consider mathematically complex ideas with real-world implications fundamentally changes students' orientations towards mathematics. In North America, such features of mathematical learning environments are rare and classroom environments more typically involve a teacher presenting examples while students are expected to sit quietly, watch, and listen, before practicing similar problems (Boaler, 2015; Hibert & Stigler, 2000). A critical feature of active mathematical engagement that has gained recognition in recent years is the opportunity for student agency (Boaler, 2002b; Gutstein, 2003) – when students get the opportunity to express their own ideas and combine their own thinking with standard mathematics. Students who use their own ideas alongside and connected through standard mathematical methods engage in what Pickering has referred to as a 'dance of agency' (Pickering, 1995). Studies have shown that active engagement with mathematics increases student interest (Boaler & Greeno, 2000; Engle & Conant, 2002; Martin, 2009) as well as high achievement and persistence in the discipline (Boaler & Staples, 2008). Such classrooms also offer students opportunities to engage with authentic mathematical work, rather than simply rehearse procedures that they may never need or use again.

While some classrooms in the United States offer students opportunities to solve complex problems and to act with agency in using and adapting mathematical methods (e.g., Ball, 1993; Lampert, 2001; Maher & Martino, 1996), such classrooms are rare and often such learning opportunities are restricted to high-level courses. When teachers inherit a class of students who have been identified as low achievers or "strugglers," they often assume that procedural, low-level remediation is most appropriate (Anyon, 1980, 1981; Haberman, 1991). In this article, we consider a teaching approach that took the opposite approach. As we set out in the first paper, a diverse, heterogeneous group of students, many of whom had persistently failed mathematics classes, were invited to solve complex mathematical problems, act with agency, and reason about mathematics through peer collaborations. Our goal in this paper is to increase understandings of the ways such environments may impact students' engagement and understanding of mathematics.

Engle and Conant (2002) describe four features of 'disciplinary engagement' in these ways:

(1) Problematize content – providing students the opportunity to question, be curious, and conduct their own inquiries;
(2) Enable student authority – encouraging students to take an active role in defining, addressing and resolving problems;
(3) Hold students accountable to others and to disciplinary norms – encouraging students to listen to each other and to seek reasons for explanations that are accountable to the norms of the discipline;
(4) Provide relevant resources – giving students access to resources such as time and materials, which enable the first three features (p. 406).

The four features Engle and Conant describe were central to the teaching intervention and associated research presented here. Like Engle and Conant (2002) our goal in this paper is not to promote a particular teaching intervention but rather to contribute to increased understandings of the different ways "productive disciplinary engagement may be fostered" (p. 401). Amidst concerns about the number of U.S. students failing algebra, and the move to introduce algebra to younger, mathematically underprepared students, we present a case of transformative student engagement that might otherwise be difficult to locate in the current educational climate.

In examining the intervention designed we ask: How might a five-week teaching intervention that focuses on collaboration and agency, promote student achievement and engagement in mathematics? We were motivated by the belief that by focusing on the development of mathematical 'practices' (Common Core State Standards Initiative (CCSS), 2010; RAND, 2002) through challenging collaborative tasks, disaffected students in mathematics could be re-engaged.

2. Theoretical framework

In mathematics education research, two theoretical developments of recent years have converged to powerfully highlight the need for studying student engagement through mathematical activity. Situated theories of learning moved researchers of mathematics education from the constructivist paradigm in which they had been immersed for many years (Zevenbergen, 1996) to a new paradigm that focused on the ways students engaged in mathematics classrooms, the practices in which they took part, and the forms of participation offered to them (Lave, 1988; Lave & Wenger, 1991). Lerman (2000) described the shift from a study of cognitive pathways to the ways students work in classrooms as a 'social turn,' and it came, in part, from the realization that students' mathematical capabilities in different real world situations drew partly from their cognitive development but also from the practices in which they had engaged in classrooms. Boaler (2002a), for example, found that when students engaged in a range of problem solving practices such as choosing, adapting and applying known and invented methods, they were better prepared to solve complex real world problems than those who had been taught the same mathematical content but without engaging in problem solving practices. Researchers of recent years have come to realize that studying mathematics learning involves observing and exploring the ways students engage in mathematics classrooms.

In a parallel shift to that from cognitive to situated theories of learning, researchers of mathematics education have started to highlight the importance of what have been named 'mathematical practices' (Boaler, 2002c; CCSS, 2010; RAND, 2002) as set out in the first paper. The new Common Core State Standards in the US include a section on mathematical practices that require all teachers to pay attention to these active ways of using mathematics as they work with students.

In the United States and other western countries, there is a widely held belief that students' mathematical potential is determined by their 'ability' (Boaler, 2013a; Boaler, 2013b; Dweck, 1986; Dweck, 2001), and school-wide practices, such as tracking, rest upon such beliefs (Boaler, 2015; Boaler, 2013a; Burris, Heubert, & Levin, 2006; Oakes, 1992). In contrast, this intervention presumed that teaching has the power to change students' achievement dramatically, especially when students are offered mathematically challenging work, are given opportunities to take responsibility and express authority for learning (Engle & Conant, 2002), and are engaged in mathematical practices like questioning, reasoning, and generalizing. In a seminal study, White and Frederiksen (1998) showed that teaching students learning practices in science and giving them time to reflect upon them led to remarkable changes in achievement. Specifically, the authors showed how previously "low achievers" began acting and *achieving* like "high achievers" when they became aware of the ways they should engage in science class. Inspired by this line of work, we developed a five-week exploratory algebra class that communicated high expectations to all students and taught them ways to actively engage in mathematical practices through collaboration. The details of the teaching approach were communicated in paper 1, this article will describe the research that was conducted with the students focusing upon the ways the teaching impacted the students.

3. Methods

3.1. Setting and participants

The summer school classes were divided into four groups of approximately twenty-four students who had just completed 6th or 7th grade ($n = 94$). The students were diverse, both racially and socio-economically, as well as with respect to prior academic achievement. See Paper 1, this issue, for details on student demographics and prior achievement. In a survey administered in the first days of class, with approximately half of the students ($n = 54$) students were asked why they were there and whether they wanted to be there. Only 10% of the students had chosen to attend the class, the remainder had been advised or made to attend by parents (59%), or by their teacher, school, or some combination of all three (31%). A detailed analysis of the design and enactment of the teaching intervention can be found in Paper 1, this issue.

3.2. Data sources

In order to learn about the students' prior experiences with mathematics, researchers collected grades from the Spring quarter preceding the summer classes, reviewed the summer school applications which included a section in which the teacher explained why they were recommending the student, and interviewed 35 students at the beginning of the summer.

The main focus of the research was on the ways students engaged and possibly re-engaged with mathematics in class during the summer and most of the data collection took place over the summer. Observers watched lessons every day, recording the details of how students entered the room, the teaching and learning interactions, and the responses of students. Approximately 60 h of lessons were also videotaped. In addition, students were asked about their reaction to the teaching, the aspects they found helpful or unhelpful and their prior experiences both through surveys, written reflections and interviews. The surveys were given out weekly or more regularly in some weeks. Student reflections were collected approximately twice a week and were mapped to the development of the class. For instance, the first week's questions elicited students' attitudes toward mathematics and school in general. Later, reflection questions emphasized key principles, for example, "When I get stuck do I try a strategy or ask for help?" or "Am I willing to help others if asked?" or, "How well did I listen?" Students were asked about specific strategies such as organizing their work, and practices like problem solving. Students were also asked to identify their favorite activities from the week. Thirty-six semi-structured interviews were conducted with 23 students, primarily in groups of two, throughout the summer. In the fall, 15 additional interviews with the same 23 students were conducted at the schools they were attending. Two researchers, who were not teaching in the intervention, reviewed and coded all student reflections, interviews and surveys. Coding was initially conducted independently on a subset of student responses to develop an initial set of coding themes – some of which explicitly reflected features of the intervention (e.g., collaboration) while others did not (e.g., fun, visual). The researchers then compared coding and evidence to revise, collapse or eliminate codes until 12 final codes were defined. All textual evidence from surveys and reflections were then coded and the results were disaggregated by code to examine variations within each theme. The codes included categories such as enjoyment, group work, mathematical seeing, and teacher. In this paper 5 of the codes are used as organizing themes, which draw upon all of the analyses of conducted reflections, interviews and surveys.

We had not intended originally to collect achievement data from the students as the students had come from 35 different middle school classes and had been learning different mathematics. But during the fifth and last week of the summer class the teachers learned that all students had taken the same assessment in the end of the spring quarter – the Mathematics Assessment Resource Service (MARS) assessment. The MARS assessment focuses on grade-appropriate mathematics content. MARS assessments are aligned with other standardized multiple-choice assessments such as the California Standards Test (CST), but differ in that they ask students to explain their thinking in writing. In the year of the teaching intervention one-fifth of questions on the 6th grade test and one-fourth of questions on the 7th grade test assessed algebra. At the end of the summer students were given the MARS algebra questions that they had taken previously. This was not ideal as the MARS algebra test assessed some content from the summer, such as extending patterns and using variables, but it also assessed content not addressed, such as solving equations with two variables. But this data did provide a pre and post-test of algebraic content for the students. In order to score the MARS test administered at the end of the teaching intervention, researchers were trained by a MARS assessor. Seventy-eight percent of the students ($n = 76$ of 98) took the final assessment. Scorers reached 97% inter-rater agreement on independently scored assessments.

In order to consider whether or not the teaching intervention had an impact upon achievement, we collected student grades for courses taken both prior to and subsequent to the summer program. In the fall, students from the summer program returned to 33 different math classes in their regular middle schools under the titles *Math 7*, *Math 8*, and *Pre-Algebra*. All three courses covered content fundamental to algebraic reasoning. To examine the potential impact of the intervention on students' performance in the subsequent academic year (fall and winter terms), math course grades were obtained for all students attending summer school ($n = 424$) whether or not they were in the 4 math classes. This allowed for a statistical analysis of grades comparing students from the teaching intervention ($n = 94$) with those in summer school but *not* in the intervention ($n = 330$). The students attending summer school but not in the math classes were a very suitable comparison group as their math achievement was equal to those attending math classes and they had all received a five week summer teaching course. Although student grades are a non-standard measure, they can provide some insight into both achievement and engagement within a learning community. In order to consider whether the acts of working we had encouraged among students were being used by students in the fall following the summer intervention, researchers observed 8 of the students' mathematics classes in their school districts in the fall.

4. Results

4.1. Pre-post test achievement outcomes

Fig. 1 depicts the mean scores for students on the algebra portions of the MARS assessments used as pre and post-tests. The pre-test mean was 48.8% while the post-test mean was 63.0%. Following the teaching intervention there was a 24% increase in students' average scores and a one-way ANOVA revealed this as statistically significant ($F(1,84) = 6.09$, $\rho = 0.016$).

Given that we had not intended to assess improvement on content knowledge and the assessment did not mirror the content we taught, the significant improvement in core algebraic content was seen as a particularly positive result. However the main aim of our teaching was to *engage* students differently in math class and we turn now to an analysis of the students' changed engagement. The analysis that follows is organized by the themes that emerged most strongly from the data.

4.2. Introduction

A review of summer school applications showed that most teachers referred students to the program because they were regarded as weak at math or were overly concerned with socializing. The students' initial participation in the classes reflected their less than positive enthusiasm about attending class. In the first session, students clearly indicated their preference for socializing and avoiding mathematics by chatting loudly with friends and resisting teacher requests. Other students were more quietly resistant – watching events without seeming to get involved, and still others displayed a lack of confidence and reluctance to engage publicly by shying away from teacher questions. Some of the students who had reached higher levels of achievement previously seemed excited to do mathematics, even if a bit reluctant to take on the group-based format. But as the summer progressed, students' dispositions began to shift and engagement spread throughout the classes. Students seemed progressively more interested and happy to take up opportunities to ask questions, discuss mathematics openly and to reason mathematically. At the end of the summer students were asked in a survey 'how much have you enjoyed this

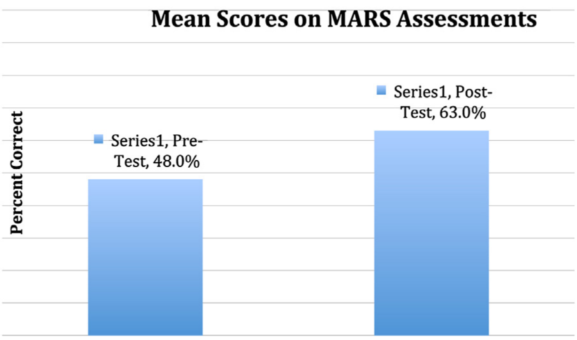

Fig. 1. Comparison of students ($n = 43$) pre and post-test results on algebraic portions of MARS assessments.

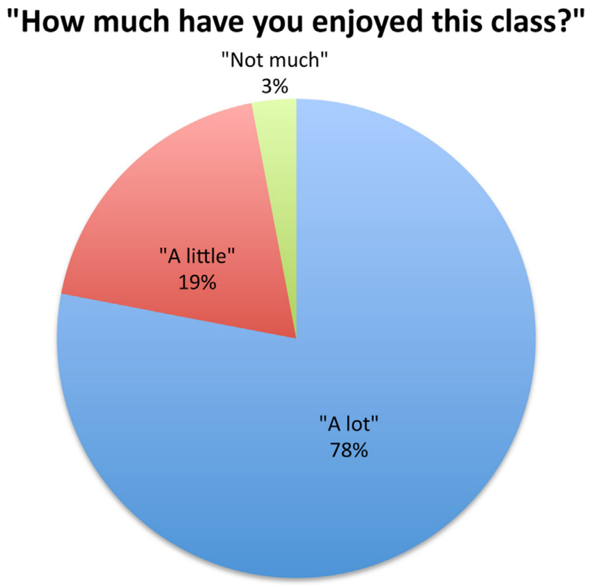

Fig. 2. Student enjoyment of summer school math class ($n = 74$).

class?' choosing between 'a lot,' 'a little' and 'not much' (see Fig. 2). Seventy-eight percent of students chose 'a lot,' 19% chose 'a little' and 3% chose 'not much' ($n = 74$).

In the same survey 87% of students said that the summer classes were more useful than their regular math classes, when asked which classes had been more useful ($n = 70$) (Fig. 3).

4.3. Collaboration and engagement

Through analysis of student interviews and classroom observations it emerged that collaboration was the most powerful aspect of the teaching intervention for students, and one that changed students' engagement and interest in mathematics. Fifty interviews were conducted with 23 students from the 4 classes and collaboration was emphasized by 20 of the 23 students as critical to their engagement. In addition to the frequency with which students reported collaboration as *the* most important part of the class, it was also the reason students typically gave first and most emphatically when explaining

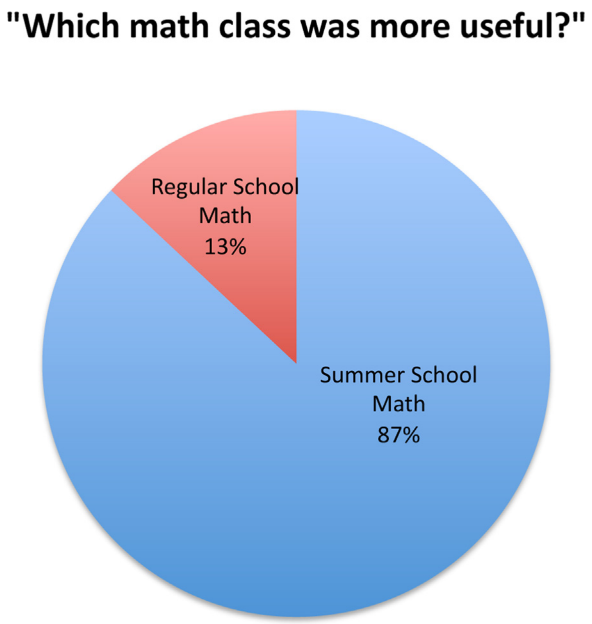

Fig. 3. Relative utility of summer and regular school math classes ($n = 70$).

how the summer intervention was a positive contrast to their prior mathematics learning experiences. The two main reasons students gave for the importance of collaboration were that it increased their interest, and it was an opportunity to deepen their understanding through learning from peers.

Many of the students interviewed in the summer described their previous mathematics classes as silent places. Representative of many students, Tanya remarked:

> For the past year, math year was the hardest because you're not supposed to talk you're not supposed to communicate. . .That's a good class to [the teacher].

Another student, Alonzo similarly observed:

> In other [math] classes it used to like be hard doing my work cause it used to be so boring. . .and I used to get frustrated and stuff and like right here we get to do group work and we get to talk and stuff and that like helps it not be so boring.

In addition to the students who conveyed the importance of discussions for enjoyment, many of the students talked about the value of learning from each other's thinking. Part of the teaching intervention had involved student presentations. Some of the students, understandably, found the idea of presenting work very difficult, and continued to discuss the difficulty of class presentations even at the end of the intervention, for example: "I don't like [to] share in front of the class because I don't like talking in front of the class but I tell my group my ideas." Other students, however, expressed a sense of confidence developed *through* presenting work:

> I didn't like presenting my work in front of the class because it was hard to explain other work to other people and being in front. Now I am not shy about going up in front of the class.

The most striking example of changed participation came from Erica, who had been characterized by her previous teachers on the referral form as a 'selective mute' and who started the class talking only to her close friend. By the end of the class Erica was volunteering to present ideas to the whole class. In interviews she explained that she had gained the confidence to present publicly because the class valued different methods and perspectives in the solving of problems and was not all about right and wrong answers. Erica had gained an A in her previous math class but wrote in her journal, "I used to use only one way the teacher taught me and not really understand it. Now I use different ways until I get it." Giving students the space to experiment and learn from one another was valuable to Erica and others. As Thomas described, "Presenting your work is very useful because it gives the students a chance to present good work to the class and gives other kids the chance to see what they did wrong." The vast majority of the students interviewed referred to group conversations as enjoyable and as avenues for learning from multiple perspectives. As Jamie reflected, the class "showed me that in life we can listen to other people's ideas and to share yours with people."

Most of the students' attending summer school had learned mathematics in classes in which they had been expected to work in silence. For these students the opportunity to talk about mathematics had been transformative, not only for their engagement but for their understanding. A few students came from mathematics classes in which they were encouraged to talk, but these students also appreciated the different ways in which collaboration was encouraged in summer school, highlighting the fact that the teachers valued multiple methods and ideas, which, as Erica said, enabled her to feel comfortable presenting ideas, even at the front of the classroom to the whole class.

4.4. Mathematical agency

One of the major goals driving the design of the teaching intervention was to give students the opportunity to engage actively with mathematics and to act with agency in choosing, using and adapting mathematical methods. Though the students did not use the term 'agency,' they spoke repeatedly about acts of choice that were mathematical in nature. The most prominent comments were separated into three areas: (1) multiple methods, (2) mathematical seeing, and (3) mathematical tinkering.

4.4.1. Multiple methods

As previously discussed, the tasks and activities chosen were meant to promote student agency by allowing students to select their own methods, representations and pathways through problems. In whole-class discussions, for example, students were regularly asked to present their methods to peers, they were also asked to color-code images to trace their methods, and to represent and illustrate mathematical problems as images. In surveys students frequently cited the opportunity to choose methods and adapt problems as important and in interviews 15 of the 23 students chose to highlight the importance of mathematical agency to their increased engagement. Alongside the value of acting with agency, the students recognized the *authority* that was given to them as they did so (Engle & Conant, 2002) as they were invited to decide upon the validity of methods, rather than simply follow a teacher's directions. As Nicole highlighted: "When I don't know how to solve a problem the way the teacher does it, I have other ways to solve it." When we interviewed John in the fall after summer school he reflected that:

> When there's a problem and people think like there is only one way, but like when I went to summer school I had like a couple of different ways to solve it, to mix it around.

The fact that students saw, often for the first time, that there is more than one way to solve problems was clearly important for their changed engagement and their opportunity to work with agency. Sengupta-Irving (this issue) gives a more detailed analysis of the students' opportunities to work with agency, and the impact it had upon them.

4.4.2. Mathematical seeing

Many of the students' reports reflected their appreciation of being able to *see* mathematical problems and ideas in different ways. Josiah, for example, told us "When we would see the problem in different ways we would understand it better." In final interviews some students remarked they had never "seen" a mathematical idea before, which speaks to the contrast between the students' summer school and regular school mathematics experiences. Tanya captured the importance of this mathematical perspective eloquently, when reflecting upon the difference between the summer algebra class and her regular math class:

> It's like the way – the way our schools did it is like very black and white, and the way people do it here it's like very colorful, very bright. You have very different varieties you're looking at. You can look at it one way, turn your head, and all of a sudden you see a whole different picture.

The notion of mathematical seeing emerged from our data as students reflected on the ways that mathematical agency and authority helped them to gain insights into mathematical ideas and to literally 'see' them, which was an extremely important resource for them. Bridging the idea of multiple methods and its relationship to mathematical seeing, Alonzo, who wrote on the importance of this across three surveys noted:

> It doesn't matter if your work is not the same, it can be right but solved differently. . .color code and be open to other strategies. . .be open-minded and have a better understanding of other methods and strategies.

Many students reported that they developed more open mathematical dispositions, that included a new form of mathematical understanding and agency – that of *seeing* mathematical ideas.

4.4.3. Mathematical tinkering

A third important part of the intervention for students involved the manipulation of problems as students were encouraged to adapt and extend problems, as they worked. Many of the students reflected upon the opportunity to change or create mathematical avenues, rather than simply manipulating methods to recreate known answers. Louisa, for example, when asked what she had learned about algebra and patterns wrote, "I have learned that after finding a pattern you can stretch it in many ways instead of just staring at it. I have learned to think beyond the answer to the problem." Ivan also talked about *going beyond*, when he said, "When I'm done, I think of something harder to do." Nancy spoke similarly when she explained that generalization in particular "helped [her] to look beyond the problems and make challenges for [herself]." We have named this aspect of their mathematical work 'tinkering' to capture the spirit of the students' comments. Allowing students to tinker with mathematical ideas requires the most precious of classroom resources – time – and Jorge reflected on this saying: "[In the summer] we stay on [tasks] longer. . .so we can really get to know how to do pattern blocks and everything, and try to figure out the pattern." Jorge, a previously low achieving student, who had been expelled from two schools, been placed in low tracks and whose previous math grade was an F, gave a striking answer when asked what advice he would give to his teachers. He simply said that they should "give harder problems." Jorge had enjoyed working on complex and open mathematics problems with his peers, that challenged him and that he could develop into even harder problems. Jorge's engagement changed dramatically in summer school, something Selling (this issue) analyses in depth. For a student who had probably been given low-level work without opportunities for agency or responsibility, being given the chance to discuss, investigate and extend mathematics problems was transformative.

In discussing the opportunities students received to adapt and extend problems, working with standard mathematical methods at the same time as using their own ideas, the students were engaging in what Pickering (1995) refers to as the 'dance of agency.' Pickering (1995) has studied the work of mathematicians as they move the discipline forward with new theories, and noted that they always engage in this 'dance' as they move between using their own ideas and using the standard methods of the discipline. Instead of talking about the ways that a mathematical dance enabled new mathematical discoveries, however, these students were discussing the ways that such flexibility and tinkering enabled their learning. They spoke about the value of stretching problems and of simply having the opportunity to use their own thoughts and ideas, which was a new experience for most. As Sophia noted, "In a regular class you have to do everything the teacher's way and sometimes I don't get it and in these classes I do the problems in whatever way I want." Tanya, reflected on the opportunities for agency in particularly interesting ways:

> It was much funner in [summer school math]. You not only got to like just see the problem, you got to think it. . .you got to smell it, you got to eat it. And then after you finish the task that's given to you, you need to have another assignment to be like, well what if this changed, and you did this with that, instead of, you know, that. It just, it just opens your mind, and makes it harder, a new way of thinking.

A particularly notable and illustrative example of "tinkering" came from a mathematically reluctant student named Alonzo who was working on a task called "Staircases." In the original problem students had been asked to determine the total number of blocks in a staircase of height *n*. Groups were given scratch paper and linking cubes to explore patterns emerging from

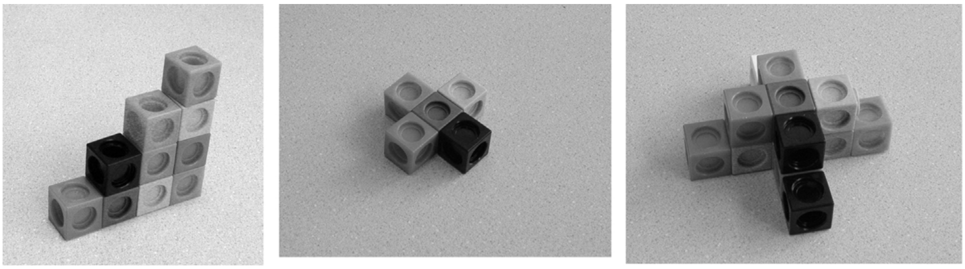

Fig. 4. Aerial view of the original Staircase (left) and the Alonzo Staircase (right).

staircases of increasing height. Having completed the given task, Alonzo created a far more involved staircase structure by extending it in four directions (see Fig. 4). Soon his classmates gathered near and several groups started exploring the "Alonzo Staircase." Some struggled with representing the models they built while others struggled with generalizing a solution for the *n*th case. In all cases the struggle was helpful for students learning (Boaler, 2016). Building the structure and determining its solution for height *n* was not enough, however. Students had to explain how the new solution related to the original solution. In this way, what was initiated through tinkering was reframed as a question requiring additional inquiry into the mathematical relationship between patterns and their physical or symbolic (and algebraic) origins.

Mariotti and Fischbein (1997) have noted that deep mathematical understanding is enabled when there is a greater transparency in the choices that lie behind the mathematical definitions that students are often asked to accept as given. Mathematics differs from science as it explores a created, rather than physical world, but students are rarely given the opportunity to see or even appreciate the choices that determine the mathematical world. As Sfard and Linchveski (1994) noted, for the student in the algebra classroom, when the algebraic object is accepted too soon it becomes fixed in the student's mind and she or he no longer tinkers or tweaks it, as would happen in a more productive mathematical setting. Pickering identified the importance of agency for mathematical advances and conceptual thinking, but our data analysis showed that mathematical agency gave the students important opportunities for *learning*. This came about partly because students felt their ideas were used and valued, which is rare in typical math classes (see Boaler & Greeno, 2000; Gutstein, 2007) but may be critical for learning, particularly among adolescents. Students welcomed the opportunity to act with authority (Engle & Conant, 2002), to make mathematical decisions, and to be considered capable of hard, mathematical work. Gutstein (2007) taught with the goal of encouraging student agency, but specifically encouraged students to use mathematics to critique injustices and, more generally, the social order (Friere, 1970). Our goals related more to the ways students saw and engaged with mathematics, as a discipline, and to the opportunities that working with mathematical agency gave them for their learning. See Sengupta-Irving, this issue, for a detailed illustration of the role that agency played in the summer school classrooms.

4.5. The role of mathematical practices in engagement and achievement

As students worked on exploratory algebra activities, extending patterns and generalizing them, they were encouraged to engage in mathematical practices, including organizing and representing ideas in different ways – through words, pictures, tables, and so on. It emerged that the students who had achieved D and F grades in their math classes were particularly in need of help with organizing work, a critical first step in working with open-ended problems. On one of the first days of camp we gave the students the chessboard problem and observed an interesting phenomenon. The problem seemed to separate the previously low and high achieving students in their success on the problem. What separated the students was not their mathematical thinking – but the organization of the previously low achieving students, who identified and collected the different sized squares but did not keep careful records of the many different sizes and numbers. This was useful information for the teachers as we then taught all of the students – high and low achieving – how to organize their thinking. This helped the previously low achieving students enormously and in later open ended problems they achieved much greater success. In the final surveys many students wrote they had learned to be systematic and organized. For example, Stevie wrote, "I learned to organize my work by making T-tables, making charts, also I learned that I should label important information in directions," while Timothy commented more generally, "I have learned to organize my work, write it all down." This latter comment may seem minor, but it reflects an important learning practice that supported the mathematical practices of exploration, generalization and representation.

Another mathematical strategy intricately tied to the practices of exploration and representation on the path to generalization, was that of starting with a simpler case. This was very important in activities such as generalizing the number of squares in an $n \times n$ chessboard. This was a challenging practice to teach as many students thought it was cheating or wrong to answer a different question than the one originally posed. Students had learned to follow a teacher's directions precisely and it took encouragement to have them adapt, change or extend problems while preserving key features in order to make

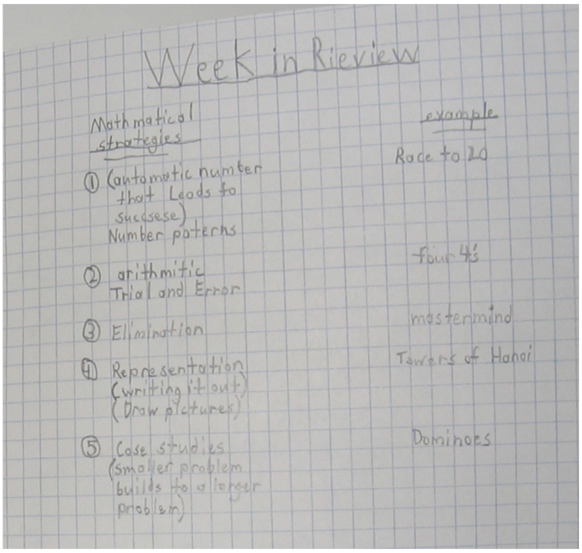

Fig. 5. Student journal entry summarizing mathematical strategies and associated tasks.

them less challenging (Pólya, 1945). In surveys, some of the students referred to this practice, Rebecca for example, reflected that,

> Patterns were very helpful because sometimes the question was asking about a huge number, so then I would just start with some smaller numbers, find a pattern and predict the answer without just taking a lot of time and effort to do the one big problem.

Others spoke of the ways that working with smaller cases or extending patterns promoted learning: "I learned to see patterns a lot better and how to understand how it gets bigger (or smaller)." A number of students also wrote about the importance of representation as a mathematical practice: "It is very useful because when you get to draw [a problem] out it's a lot easier than to visualize it in your head." Fig. 5 is an example of a student's journal in which he wrote about strategies and practices he learned through a review of one week's lessons:

The mathematical practices and supporting strategies that students learned were important to their enjoyment and success during the summer class but our broader goal in teaching the students mathematical practices was that these practices would also be useful to them in their future learning of mathematics. See Selling, this issue, for further analysis of the role of one mathematical practice, representation, for students in the summer school.

4.6. Future mathematics classes

The data from the students' experiences in the summer showed that the students had increased their content knowledge and that they had both appreciated and benefitted from experiencing a more open and multi-dimensional math than they were used to. In addition to the data we collected from the students during the summer classes, we considered the impact of the students' summer experiences on their participation and achievement when they returned to their regular math classes in the fall. In order to compare the students' grades with a comparable group of students we collected the grades of all students who had attended summer school. This included 80 students who had been in our classes and 271 who had attended summer school but not attended the math classes. This was an ideal comparison group as the non-math students had achieved the same grades as the students who attended our mathematics classes. Those who did not take summer school mathematics had received an average math grade of 74.4% in the term before the intervention while the average mathematics grade for students in the intervention was 73.3%. Moreover, students in the comparative cohort were attending the same schools as those students in the intervention. In addition to a grade comparison we observed eight of the 33 mathematics classes that students moved into in the fall and interviewed 15 of the students.

Observations of fall classes revealed students seated in rows and working alone or occasionally in pairs, with minimal or no opportunities for mathematical discussion. The questions typically posed to students by teachers required one or two-step procedures for completion. There was no evidence in observations, of the students engaging in mathematical practices like representation, generalization, or problem solving. The fall classes generally offered the types of environments in which many students had previously underachieved and about which they expressed great displeasure in interviews – silent classes in which students worked through procedures with no or little opportunity for their own thinking.

Mathematics grades were obtained for three terms: the spring term preceding the summer intervention, and the fall and winter terms immediately following it. After eliminating subjects with missing data, the sample sizes were $n = 271$ (of 330)

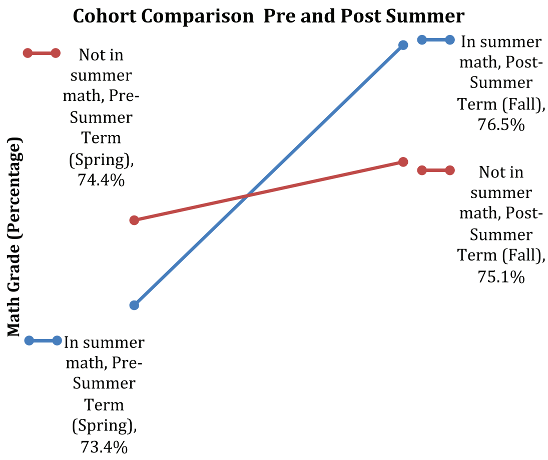

Fig. 6. Math grades for comparative (red) and intervention (blue) cohorts in the Spring (direclty preceding summer) and Fall (following).

for students enrolled in the summer program but *not* in math, and $n = 80$ (of 94) for those in math. Fig. 6 shows that despite the procedural nature of the students' fall classes, the grades of students who had attended the intervention had significantly improved ($M = 0.320$, SD = 0.1284, $t(80) = 2.23$, $\rho = 0.029$), whereas the grades of other students in the summer program had not ($M = 0.00775$, SD = 0.13901, $t(271) = 0.92$, $\rho = 0.36$). In general the students in the intervention whose grades rose most positively were those who had achieved a D or lower in the prior class, where 6th grade students improved by more than one letter grade on average, while 7th grade students improved by almost one grade on average. Four students (5%) improved their achievement by three letter grades following the summer and 27 students (34%) improved their achievement by 2 letter grades.

Fig. 6 suggests that the summer school intervention helped students with their regular math classes, even though the engagement we had taught the students was not enabled. But two quarters after the summer intervention the grades of both cohorts fell and the improvements made by students in the intervention had been erased by the winter term.

The data we collected on grade changes shows the potential of the teaching intervention for increased student engagement and achievement, even in traditional mathematics classes, while also demonstrating the limitations of a short-term teaching intervention on the long term learning of students.

5. Discussion and conclusion

When students underachieve in mathematics, a common response is often to give remedial, skills-based work and to assign them to low track classes. In the 5-week teaching intervention described here students who had failed previous mathematics classes were invited to engage actively with mathematics, and to pursue challenging algebraic problems collaboratively with peers of varied backgrounds and achievement levels. This was a change in environment for all of the students and it prompted improved engagement and learning. The intervention enabled students to work at a level closer to their mathematical potential, and the research data give some insights into the reasons for this. These related to aspects of the teaching – in particular, being able to discuss methods and to see problems from different perspectives, being able to collaborate and see mathematics as a social endeavor, and being able to work with agency with students using their own ideas as well as formal mathematical methods to solve problems. Some teachers shy away from collaboration and agency because they think students are incapable of acting responsibly, particularly when they are low achievers. Our data suggested the opposite to be true: when students were given opportunities to express authority and agency they acted extremely responsibly and appreciated challenging, difficult work through which they could show their worth.

Students in the intervention were actively engaged in learning mathematics and were provided a new way of relating to the discipline. The two most promising results in this respect were the comparative perspectives that students reported, and the fact that they achieved significantly higher grades after the intervention, even though the classes differed markedly from the intervention. It is perhaps not surprising that the students' changed engagement did not continue into the winter because of the relatively short duration of the intervention and because students returned to the types of teaching environments they had described as *causing* their disaffection – silent classrooms in which they practiced procedures. The summer teaching classes were completely heterogeneous – culturally, socially and academically and the heterogeneity was part of the success of the intervention. Students from both ends of the achievement spectrum appreciated working with students at different levels and the teachers of the classes all believed that they would have been less successful if the classes had been made up of all low achievers or even all high achievers. The different students offered different perspectives, methods and ideas in the discussion of problems and the variety in their experiences enriched the classroom conversations.

The new Common Core Mathematical Standards, which have been adopted by 49 states and territories in the US, include a critical section on 'mathematical practices,' which require active ways of working that were at the center of this teaching intervention. This new set of standards and their associated assessments provide new possibilities for classrooms in which teachers pay attention to mathematical engagement, and students actively "do" rather than simply "receive" mathematics. Research continues to show the importance of problem solving, reasoning and communicating for students' engagement in mathematics classes but until teachers can be supported in teaching students in such ways, it seems students will continue to experience frustration. As Tanya poignantly summarized when researchers interviewed her in the fall following the summer school, "the only way to describe the summer school is very colorful and the [regular school] is just still, ugh, black and white. And you wanna ask, 'Can I have a little bit of yellow?"' Tanya's request, for a 'little bit of yellow' in her mathematical landscape, seems far from unreasonable, and it may indicate an urgent direction for future research in mathematics education. For while the field has developed a sophisticated understanding of ways to engage students in mathematics over recent decades, we lack a comparable understanding of the ways teachers may be supported in offering mathematical environments that invite students to experience the many colors of mathematics.

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