

California Digital Learning Integration and Standards Guidance

Submitted by:

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Mathematics

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The following is an extract from the California Digital Integration and Standards Guidance.

Introduction

The purpose of Chapters 5-9 is to present standards and instructional guidance to support the continuum of learning from TK/K through grade 10. These chapters prioritize critical areas of instructional focus by grade levels. Attention to these critical areas will ensure that students transition to the next grade level well prepared to learn new skills and concepts. This guidance serves as a companion resource to the *California Common Core State Standards: Mathematics* and the draft 2021 *Mathematics Framework*. The organization of the content and practice standards as “big ideas” raises the individual standards to a higher level of “big ideas” and highlights the importance of the content and the ways it is connected to other content and practices. The standards guidance is intended to support teachers as they implement math instruction in online, blended, or in-person learning environments.

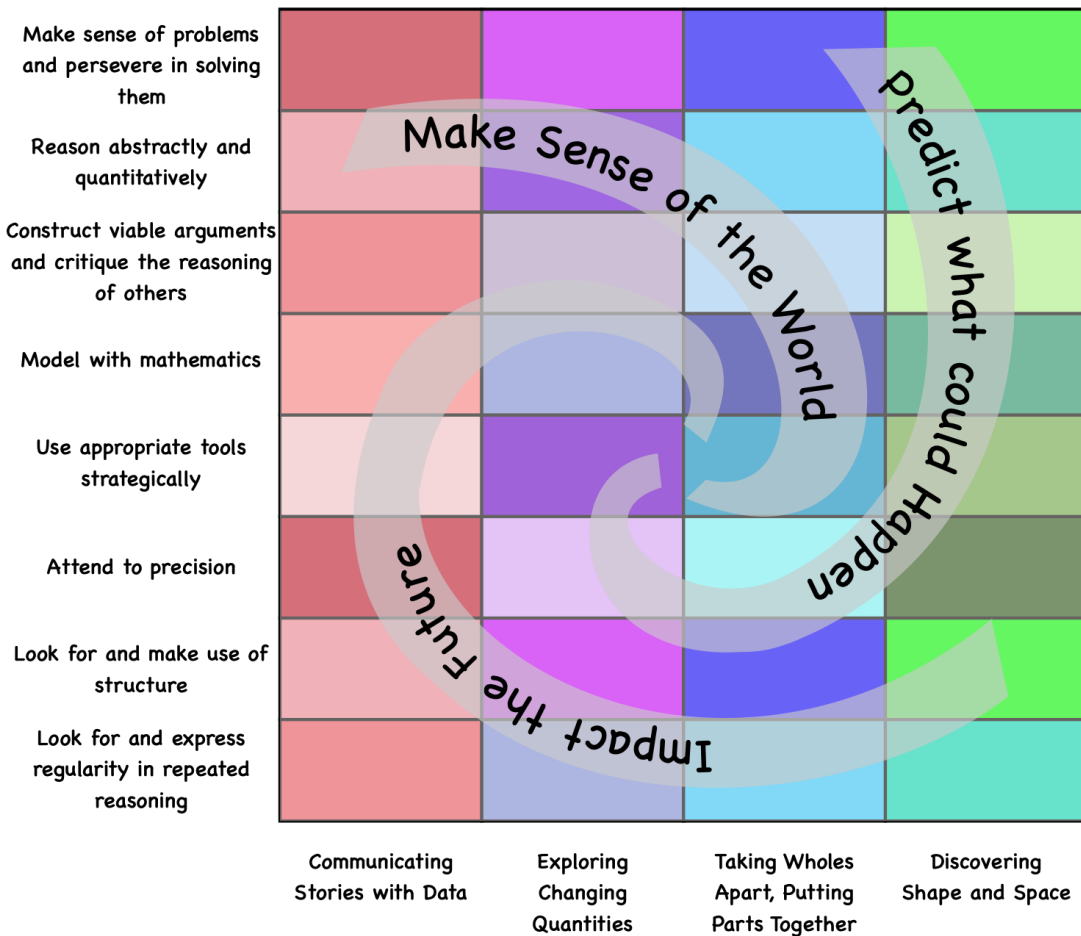
California’s goal for all students is that they learn mathematics as a meaningful subject of connected ideas. Teaching with meaning and connections requires a different organization of content and practice standards. The draft 2021 *Mathematics Framework* advocates for teaching to “big ideas” rather than organizing teaching around the small descriptions of mathematics set out in the standards. Mathematics professor Randy Charles defines a big idea as a “statement of an idea that is central to the learning of mathematics, one that links numerous mathematical understandings into a coherent whole” (Charles & Carmel, 2005, p.10). A “big idea” approach has been shown by research to engage students and increase achievement (Boaler et al., 2021; Cabana et al., 2014, Makar, 2018). The approach raises the individual standards to a higher level of “big ideas” that show teachers and students the importance of the content and the ways it is connected to other content and practices. As teachers orient their approach to big ideas and connections, they will find that there seems to be less content to teach and more time for students to explore ideas and learn deeply. The same content is actually taught and learned, but the organization of connections and big ideas allows for a more coherent approach in which students learn different, connected ideas together. The draft 2021 *Mathematics Framework* has organized Content Connections (CCs) of “Communicating stories with data,” “Exploring changing quantities,” “Taking wholes apart and putting parts together,” and “Discovering shape and space,” and this document organizes the big ideas under these broad content headings, which are explained in more detail below. Each grade band section (TK-2, 3-5, 6-8, 9-10) shows the progression of big ideas across the grades.

The draft 2021 *Mathematics Framework* includes a principle that mathematics learning in classrooms should always have a purpose and that rather than students working through questions without mathematical direction, they should work on an approach of “investigating and connecting.” To do this, the Framework recommends “crosscutting drivers of investigation” that can guide investigations. The drivers are:

- **Making sense of the world** (understand and explain),
- **Predicting what could happen** (predict), and
- **Impacting the future** (affect).

Figure 5.1 shows these drivers and the ways they can be applied to any combination of content and mathematical practices.

Figure 5.1. The Drivers of Investigation, Content Connections, & Mathematical Practices from the draft 2021 *Mathematics Framework*.



Long description: Three Drivers of Investigation (DIs) provide the “why” of learning mathematics: Making Sense of the World (Understand and Explain); Predicting What Could Happen (Predict); Impacting the Future (Affect). The DIs overlay and pair with four categories of Content Connections (CCs), which provide the “how and what” mathematics (CA-CCSSM) is to be learned in an activity: Communicating stories with data; Exploring changing quantities; Taking wholes apart, putting parts together; Discovering shape and space. The DIs work with the Standards for Mathematical Practice to propel the learning of the ideas and actions framed in the CCs in ways that are coherent, focused, and rigorous. The Standards for Mathematical Practice are: Make sense of problems and persevere in solving them; Reason abstractly and quantitatively; Construct viable arguments and critique the reasoning of others; Model with mathematics; Use appropriate tools strategically; Attend to precision; Look for and make use of structure; Look for and express regularity in repeated reasoning.

Big Ideas and Network Maps

The *California Common Core State Standards: Mathematics* offer domains, cluster headings, and standards – with most textbook publishers translating the detailed standards into short, procedural questions. A problem with working through standards and associated questions is that teachers do not have time to go in depth on any of the standards, or even to teach them all. A different approach is to consider the big ideas, as set out in the introduction to this section, that bring in many different standards, that often go across the clusters and domains. As students work on rich tasks, they will encounter many of the standards but in a more connected and meaningful way. This document sets out this “big idea” approach to mathematics, with the goal of helping teachers and their students, both during a period of decreased learning time and moving forward.

To highlight mathematical connections, each grade has a network map which shows the big ideas as nodes. These represent important and foundational content, and an ideal approach to teaching mathematics, in person or online, starts with choosing rich tasks that focus on the big ideas.¹ As students explore and investigate with the big ideas, they will likely encounter many of the different content standards and see the connections between them.

The size of the node relates to the number of connections it has with other big ideas. The connections between big ideas are made when the two connected big ideas contain one or more of the same standards. The big idea colors in the nodes correspond to the table where the big ideas are correlated with full descriptions. The descriptions of each big idea are not taken from the standards or the clusters or domains; rather, they are new descriptions, as many of the ideas go across clusters and domains. For example, in grade 3 the big idea: Fractions of Shape & Time, brings together standards from the domains of Measurement and Data, Number and Operations in Base Ten, Fractions and Geometry. The new descriptions integrate well with the mathematical practices, as they describe mathematics as a subject of reasoning and communicating.

¹ <https://www.youcubed.org/resource/k-8-curriculum/>

Chapter 6: Mathematics in Transitional Kindergarten through Grade Two

Figure 6.1. A Progression Chart of Big Ideas through Grades TK–2

Content Connections	Big Ideas: Grade TK	Big Ideas: Grade K	Big Ideas: Grade 1	Big Ideas: Grade 2
Communicating Stories with Data	Measure & Order	Sort & Describe Data	Make sense of Data	Represent Data
Communicating Stories with Data	Look for Patterns	n/a	Measuring with Objects	Measure & Compare Objects
Exploring Changing Quantities	Measure and Order	How Many?	Measuring with Objects	Dollars and cents
Exploring Changing Quantities	Count to 10	Bigger or Equal	Clocks and Time	Problem solving with measures
Exploring Changing Quantities	n/a	n/a	Equal Expressions	n/a
Exploring Changing Quantities	n/a	n/a	Reasoning about Equality	n/a
Taking Wholes Apart, Putting Parts Together	Create Patterns	Being flexible within 10	Tens and Ones	Skip Counting to 100
Taking Wholes Apart, Putting Parts Together	Look for Patterns	Place and position of numbers	n/a	Number Strategies
Taking Wholes Apart, Putting Parts Together	See and use Shapes	Model with numbers	n/a	n/a
Discovering shape and space	See and use shapes	Shapes in the world	Equal parts inside shapes	Seeing fractions in shapes
Discovering shape and space	Make and measure shapes	Making shapes from parts	n/a	Squares in an array
Discovering shape and space	Shapes in space	n/a	n/a	n/a

In the primary grades, students begin the important work of making sense of the number system, implementing SMP.2 to “Reason abstractly and quantitatively.” Students engage deeply with Content Connection 3 (CC3, Taking Wholes Apart and Putting Parts Together), as they learn to count and compare, decompose, and recompose numbers. Building on a TK

understanding that putting two groups of objects together will make a bigger group (addition), kindergarteners learn to take groups of objects apart, forming smaller groups (subtraction). Students develop meanings for addition and subtraction, as they encounter problem situations in transitional kindergarten through grade two. They expand their ability to represent problems, and they use increasingly sophisticated methods to find answers.

Big ideas of number in TK–2 include the following (Boaler, Munson, & Williams, 2020):

- Organize and count with numbers.
- Compare and order numbers on a line.
- Operate with numbers flexibly.

The big ideas of data in these early grades include the following:

- Data for understanding: What questions can be asked? What data are needed to answer it?
- Defining data: What is data? How was the data collected?
- Representing and interpreting data: What does data look like, and what does it mean?

In grades TK–2, students learn to distinguish between *categorical* (non-numerical) data and *measurement or quantitative* data. For instance, consider a set of colored blocks in the classroom. “Color” is a categorical variable that students could observe about each block. “This block is 15 centimeters long” is a measurement data point. The standards develop categorical data in grades K–3 and measurement data beginning in grade two.

Figure 6.2. Examples of Categorical and Quantitative Data

Types of Data	Examples
Quantitative (or Measurement) data	<ul style="list-style-type: none"> • Color (red, green, blue, yellow) of blocks in the class set • Species of trees on the school grounds • Ice cream flavors, such as strawberry, chocolate, and vanilla
Quantitative (or Measurement) data	<ul style="list-style-type: none"> • The temperature of different drinks • Number of pages (or weight, or height) of books in the classroom • Number of students in different classrooms

Shape and space are important parts of TK–2 since students are learning to make sense of the world around them, while noticing patterns, common shapes, along with their attributes. As students develop their understanding of plane figures, while noting sides, angles, and similarities and differences across plane figures, students move on to see that plane shapes

make up the faces of solids. The importance of shape and space is heightened by the mathematical thinking that goes into defining and describing the world, as well as students building their academic vocabulary and ability to communicate their reasoning. Patterning is another critical area. Recognizing a pattern well enough to continue it or fill in missing pieces and then generalizing the pattern is crucial to mathematical development. It is important that students focus on the unit that repeats and makes the pattern since the idea of a “unit” is used throughout TK–12 mathematics.

Students can be surrounded with a wealth of two dimensional (2-D) and three dimensional (3-D) manipulatives where they can build and create, noting the composition and decomposition of the shapes that make up the world. In an online environment, teachers can ask students to look through their area for 2-D and 3-D objects. Pebbles, stones, boxes, or other items can be stacked. Describing and noticing the shapes that make up other complex shapes is an important creative way to make sense of the space around them. Students can upload pictures of what they create and describe and classify their creations. In a purely online environment, students can use the geometry in Desmos to create 2-D and 3-D shapes. Creating 2-D images of 3-D shapes is a wonderful learning experience. TapTap Blocks is a free and fun space for building in 3-D on an Apple device. Tinkercad is another option for 3-D building.

The following interview highlights an educator who is using digital tools to help students build foundational concepts introduced in this grade span.

Voices from the Field: Lisa Nowakowski | King City Union School District | King City, CA

“Helping teachers find engaging and effective ways to teach mathematics via distance learning comes down to finding the right tools for the job,” says technology coach Lisa Nowakowski. Just as she teaches students to build on existing knowledge and skills when solving math problems, Nowakowski tailors her educational technology recommendations to teachers based on what they and their learners already know. She looks for tools that are easy to adapt to existing practices and readily enhance learning experiences without putting additional strain on teachers and students.

In her 26 years as an educator, Nowakowski has taught everything from kindergarten through fifth grade and is currently a technology coach for the King City Union School District in California’s central coast. She shares how she collaborates with teachers to integrate technology efficiently and dynamically to teach math to elementary students.

How have you been helping teachers and students thrive through distance learning?

Several years ago, I developed MathReps for my fifth graders. The idea was to help them practice and retain math skills because, by the end of the year, they would get rusty and forget concepts they learned at the beginning of the year. As I developed the lessons, which were making a huge difference with my own students, I posted them online, free for other teachers to use. When we went to distance learning, teachers have shared with me that they’ve been adapting these resources using digital tools. For example, Flipgrid [video-based discussion

software] and Jamboards [collaborative digital whiteboards] were used to allow students to not only show their work, but also explain and talk to each other about how they got their answers.

In what ways do you employ digital tools to enhance what may be done in a traditional classroom setting?

I just ran this activity in a class of first graders with a puzzle that had pictures of four dogs. They learned to articulate the similarities and differences they saw—this one had spots, or this one was tiny or extra-long, or all of these other dogs are the same height. The activity is adaptable, too, because I then leveled it up for a class of second graders, where they had to verbalize their rationale and also type it out on Nearpod's [a formative assessment platform] collaborative board. We talked it out so that, just like in a traditional classroom, if you weren't sure what to do, you could hear other examples and see how other students were thinking about it virtually.

What we're trying to do is still have those group conversations, which are so powerful and needed—and not just for developing math sense. My district has a high number of students who are English learners; it's really important for them to be talking about these things just to hear the different vocabulary and to practice speaking and listening. Having activities and dedicated space to be able to talk things out is simple, but so powerful.

What's the best way to balance synchronous and asynchronous learning experiences for students?

What we do is try and make asynchronous assignments engaging and fun—something that isn't boring and that they may already have familiarity and success with. For the synchronous work, we schedule in the higher level, harder concepts that they need more guidance through. Once they get more proficient at those skills, then those get moved to asynchronous assignments, and so on. The teachers in our district are really just trying to balance the learning experience to make it as engaging and interesting as possible.

How are you using technology to help students learn foundational math concepts?

If you're having students solve $36+45$, for example, and asking them to think about different ways to break these numbers apart, you can use Jamboard [collaborative digital whiteboards], and they can write out their reasoning, like $30+40$ and then $6+5$ on virtual sticky notes, or if they already know Google Slides, you can have them each make their own presentation or make videos on Flipgrid [video-based discussion software] to explain their logic. Just like when solving math problems, there are multiple paths to the right answer—it's really whatever tool works best for you and your students.

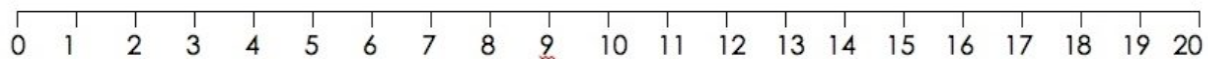
As the technology coach, people often ask me what the best tool is, and I always tell them it's really about what they are trying to do and how. We'll talk about the tools they're already using and then build from there. I don't rely on one-size-fits-all recommendations for everyone because, just like we have to be flexible with numbers, we have to be flexible with the tools we use. It's about finding the right tool for the job.

Transitional Kindergarten

The work of learning to count typically begins in the preschool years. The *California Preschool Learning Foundations, Volume 1* includes foundations in mathematics that cover five strands: Number Sense, Algebra and Functions (Classification and Patterning), Measurement, Geometry, and Mathematical Reasoning. These foundations in mathematics support the developmental progression of students from preschool through TK.²

In TK, students are working out what numbers mean and how numbers connect to fingers, objects, movement, and each other. Students learn to count objects meaningfully by touching objects one-by-one as they name the quantities, recognizing that the total quantity is identified by the name of the last object counted (cardinality). As students compare numbers, they will later be able to locate them on a line. Number lines are really helpful for students' learning and have even been found to eliminate differences in numerical reasoning between middle income and lower income students in preschool (Ramani & Siegler, 2008).

Figure 6.3. A Number Line



In TK, kindergarten, and first grade, a more accessible model is a number path. Whereas a number line shows numbers in terms of measurement, a number path is a counting model, which shows numbers as rectangles.

Figure 6.4. A Number Path



When young students count on a number line, they can miss the numbers and land on the spaces, whereas a number path allows students to count the rectangles.³

In TK, students start to compare data and numbers using objects and learn relational vocabulary, such as *more*, *fewer*, *less*, *same as*, *greater than*, *less than*, and *more than*. Dot card number talks (see Figure 6.4) are an ideal activity for students to learn to subitize, identifying a small group, in this case dots, without counting. Activities can be designed in ways that provide students with a variety of structures to practice, engage with, and eventually master the

² <https://www.cde.ca.gov/sp/cd/re/documents/preschoollf.pdf>

³ <https://www.tapfun.com/blog/number-paths-a-fabulous-tool-for-kindergarten-and-first-grade-math#:~:text=A%20number%20path%20is%20a,confidence%20and%20accurately%20solve%20problems>

vocabulary. In TK, students learn to distinguish between *categorical* (non-numerical) data, such as color, and *measurement* or *quantitative* data, such as the height of a plant.

Young children love to build and create. Students can be encouraged to develop creative scenes using 2-D shapes, as well as create linear patterns and arrays of shapes, including composing simple shapes to form complex ones. An important concept for students to learn is that shapes can transform in space and maintain their congruence. For example, a triangle can spin and flip, but it is still the same triangle just oriented in space differently. Allow students time and space to play with shapes. They can be encouraged to describe a shape's position in space compared to other shapes and start to use language that describes similarities and differences in shapes, as well as magnitude, direction, and distance.

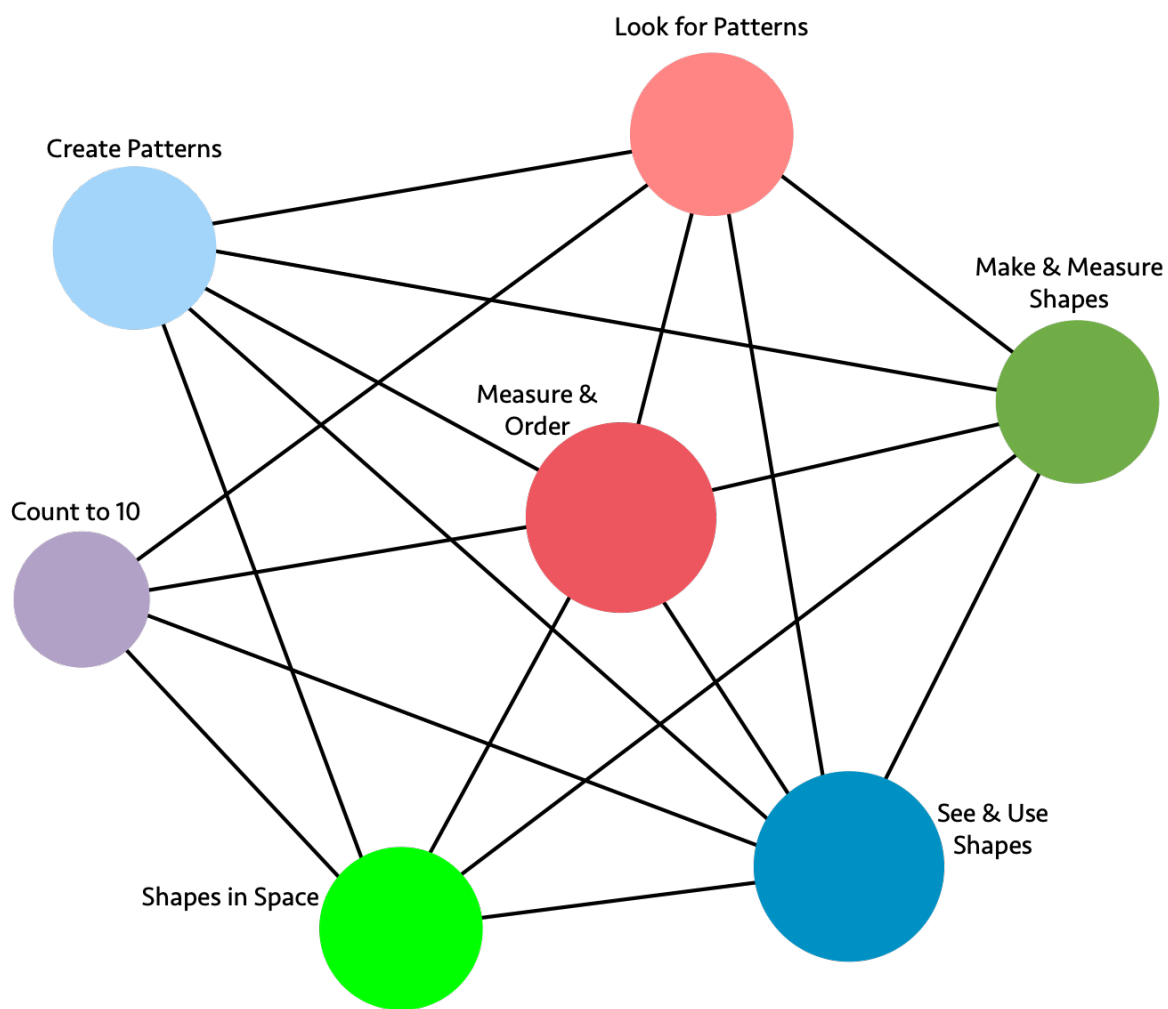
Patterns are a natural beginning to mathematical thinking for young children. This is a time where seeing patterns supports a young learner in making sense of their world. Students can be encouraged to notice patterns in everything they experience, in school and at home, and can be encouraged to describe and communicate the attributes they see and the ways they see the patterns. Students can be asked to describe what would come next, solidifying that they have recognized a pattern. The physical act of building or continuing the pattern is the next important piece in their growth, culminating in their ability to communicate a generalized statement about the pattern. Students who are English learners are encouraged to use their developing English and native language assets and draw on their prior knowledge. Teachers can provide purposefully planned and "just-in-time" scaffolds and supports to engage EL students in sustained mathematical oral discourse in multiple contexts to build academic vocabulary and knowledge.

TK instruction can create rich, effective discussion where students use developing skills to clarify, inform, question, and eventually employ these conversational behaviors without direct prompting. Such instruction supports all students, including EL students, and ensures all learners develop both mathematics content and language. TK students can compare collections of small objects as they play fair share games, deciding who has more; by lining up the two collections side by side, children can make sense of the question and practice the relevant vocabulary. As the students develop understanding in recognizing numerals, they can play games with cards. The use of fingers is particularly important for students of this age, as they can represent early ideas of a number line. For a range of finger numerical activities, see YouCubed.⁴

Critical Areas of Instructional Focus

Figure 6.5. Grade TK Big Ideas

⁴ <https://www.youcubed.org/resource/visual-mathematics/>



Long description: The graphic illustrates the connections and relationships of some transitional-kindergarten mathematics concepts. Direct connections include:

- Look for Patterns directly connects to: Create Patterns, Count to 10, Measure & Order, See & Use Shapes, Make & Measure Shapes
- Make & Measure Shapes directly connects to: Look for Patterns, Create Patterns, Measure & Order, Shapes in Space, See & Use Shapes
- See & Use Shapes directly connects to: Make & Measure Shapes, Look for Patterns, Measure & Order, Create Patterns, Count to 10, Shapes in Space

- Shapes in Space directly connects to: See & Use Shapes, Make & Measure Shapes, Measure & Order, Create Patterns, Count to 10
- Count to 10 directly connects to: Shapes in Space, See & Use Shapes, Measure & Order, Look for Patterns
- Create Patterns directly connects to: Look for Patterns, Make & Measure Shapes, See & Use Shapes, Measure & Order, Shapes in Space
- Measure & Order directly connects to: Look for Patterns, Make & Measure Shapes, See & Use Shapes, Shapes in Space, Count to 10, Create Patterns

Figure 6.5a. Grade TK Content Connections, Big Ideas, and Standards

Content Connection	Big Idea	TK Standards
Communicating Stories with Data & Exploring Changing Quantities	Measure and Order	AF1.1, M1.1, M1.2, M1.3, NS2.1, NS2.3, NS1.3, G 1.1, G2.1 NS1.4, NS1.5, MR1.1, NS1.1, NS1.2: Compare, order, count, and measure objects in the world. Learn to work out the number of objects by grouping and recognize up to 4 objects without counting.
Communicating Stories with Data & Taking Wholes Apart, Putting Parts Together	Look for patterns	AF2.1, AF2.2: NS1.3, NS1.4, NS1.5, NS2.1, NS2.3, G1.1, M1.2: Recognize and duplicate patterns - understand the core unit in a repeating pattern. Notice size differences in similar shapes.
Exploring Changing Quantities	Count to 10	NS1.4, MR1.1, AF1.1, NS2.2: Count up to 10 using one to one correspondence. Know that adding or taking away 1 makes the group larger or smaller by 1.
Taking Wholes Apart, Putting Parts Together	Create patterns	AF2.2, AF2.1, M1.2, G1.1, G1.2, G2.1: Create patterns - using claps, signs, blocks, shapes. Use similar shapes to make a pattern and identify size differences in the patterns.
Taking Wholes Apart, Putting Parts Together & Discovering Shape and Space	See and use shapes	G1.1, G1.2, NS2.3, NS1.4, MR1.1: Combine different shapes to create a picture or design & recognize individual shapes, identifying how many shapes there are.
Discovering Shape and Space	Make and measure shapes	G1.1, M1.1, M1.2, NS1.4: Create and measure different shapes. Identify size differences in similar shapes.

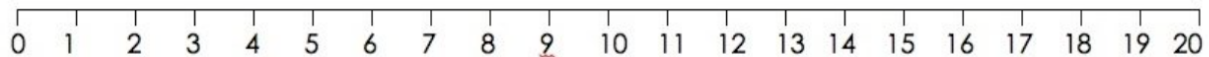
Content Connection	Big Idea	TK Standards
Discovering Shape and Space	Shapes in space	G2.1, M1.1, MR1.1: Visualize shapes and solids (2-D and 3-D) in different positions, including nesting shapes, and learn to describe direction, distance, and location in space.

Figure 6.5a includes Preschool Foundations in mathematics for students at around 60 months of age. The related kindergarten standards for TK are identified in the next section.

Kindergarten

In kindergarten, instructional time focuses on two critical areas: (1) representing and comparing whole numbers, initially with sets of objects, and (2) describing shapes and space. In kindergarten, as in TK, students are working out what numbers mean – how numbers connect to fingers, objects, movement, and each other. As students compare numbers, they will later be able to locate them on a line. Number lines are really helpful for students’ learning and have even been found to eliminate differences in numerical reasoning between middle income and lower income students in preschool (Ramani & Siegler, 2008).

Figure 6.6. A Number Line



In kindergarten and first grade, a more accessible model is a number path. Whereas a number line shows numbers in terms of measurement, a number path is a counting model, which shows numbers as rectangles.⁵

Figure 6.7. A Number Path



When young students count on a number line, they can miss the numbers and land on the spaces, whereas a number path allows students to count the rectangles.

Number talks are a particularly effective way for students to learn to compose and decompose numbers. In kindergarten, children become familiar with numbers from 1–20, and they count quantities up through 10 accurately when presented in various configurations. The use of fingers is particularly important for students of this age, as they can represent early ideas of a number line. Dot card number talks (see Figure 6.7) are an ideal activity for students to learn to

⁵ <https://www.tapfun.com/blog/number-paths-a-fabulous-tool-for-kindergarten-and-first-grade-math#:~:text=A%20number%20path%20is%20a,confidence%20and%20accurately%20solve%20problems>

subitize, identifying a group of dots without counting. As students begin seeing groups of dots as a quantity without the need for counting, they are able to partition larger groups of dots in known subitized groups, forming an important part of their number flexibility journey. Of particular importance is how numbers (and the objects they represent) and shapes can be put together and taken apart to create something new, but related. These are important ideas for the area: Taking wholes apart, putting parts together. These are powerful early steps in encouraging students to look for and name mathematical connections. As students engage in number sense explorations, activities, and games, they develop the capacity to reason abstractly and quantitatively (SMP.2) and model mathematical situations symbolically and with words (SMP.4).

As kindergarten students consider “Which has more?” questions, they can work with data, asking questions, such as “I wonder which shape has more sides?” and “Which kind of block is heaviest?” In addition to questions that can be answered with a single value, students can start to pose statistical investigative questions that involve multiple variables, such as “I wonder if plants grow more with additional sunlight?” or “I wonder if age affects which color people like?” Across the learning of different mathematical areas, students can be encouraged to use words and drawings to make convincing arguments to justify work. Students who are English learners are encouraged to use their developing English and native language assets and draw on their prior knowledge. Teachers can provide purposefully planned and “just-in-time” scaffolds and supports to engage EL students in sustained mathematical oral discourse in multiple contexts to build academic vocabulary and knowledge.

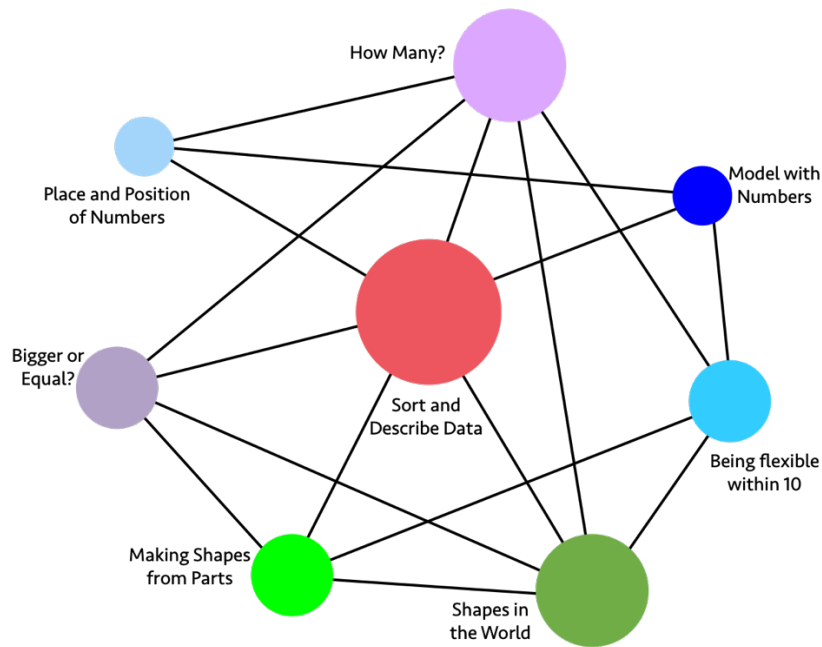
Students in kindergarten continue their exploration of geometric shapes by noticing similarities and differences in the shapes. Students can use the geometry in Desmos to create 2-D and 3-D shapes. Creating 2-D images of 3-D shapes is a wonderful learning experience. TapTap Blocks is a free space for building in 3-D on an Apple device. Tinkercad is another good option for 3-D building. When students are initially allowed to use their own words and engage with others, their use of academic vocabulary increases as they learn to describe these similarities and differences. Shapes can be beautifully connected with categorical data as students organize shapes that are squares, triangles, and circles, as well as numerical data as they note which shapes have 3 sides or 4 angles, for example. Sorting activities support students’ growth in mathematics, especially when students are given sets of objects where they, themselves, determine the categorical or numerical variables and communicate their reasoning to others. As students sort and label the attributes, they are also pattern seeking. Three dimensional shapes, solids, can be introduced, and students again can be asked to sort sets of 2-D and 3-D objects. Through this activity, they will notice that the 3-D shape faces are similar to the 2-D shapes, as composing and decomposing shapes allows students opportunities to see shapes within shapes. As students progress to seeing the relevance of 2-D shapes within 3-D shapes, they can be encouraged to combine different 3-D shapes, composing more complex shapes. It is important to include composing and decomposing shapes so students can see shapes within shapes.

Patterns are an important part of all grade levels, especially in the primary grades, as pattern seeking is the essence of mathematics (Devlin, 1996). Minds seek patterns to make sense of the

world. As students work with AB and ABA patterns and more, they are forming an important knowledge set. Attention can be paid to the repetitive unit. While it is important to fill in the gaps in a pattern or predict what comes next, careful attention can be paid to the set of items that form the base unit of the pattern. For example, a pattern where students are asked to fill in the blank (e.g., square, triangle, square, ?, square, triangle) should include a conversation about the unit that repeats. Students note that “square, triangle” is the unit that repeats itself. Pattern exploration can extend to students’ homes and lives as they learn to see and explore patterns all around them.

Critical Areas of Instructional Focus

Figure 6.8. Grade K Big Ideas



Long description: The graphic illustrates the connections and relationships of some kindergarten mathematics concepts. Direct connections include:

- How Many directly connects to: Being flexible within 10, Shapes in the World, Sort and Describe Data, Bigger or Equal, Place and Position of Numbers
- Model with Numbers directly connects to: Being flexible within 10, Sort and Describe Data, Place and Position of Numbers

- Being Flexible within 10 directly connects to: Model with Numbers, How Many, Making Shapes from Parts, Shapes in the World
- Shapes in the World directly connects to: Being flexible within 10, How Many, Sort and Describe Data, Bigger or Equal, Making Shapes from Parts
- Making Shapes from Parts directly connects to: Shapes in the World, Being flexible within 10, Sort and Describe Data, Bigger or Equal
- Bigger or Equal directly connects to: Making Shapes from Parts, Shapes in the World, Sort and Describe Data, How Many
- Place and Position of Numbers directly connects to: How Many, Model with Numbers, Sort and Describe Data
- Sort and Describe Data directly connects to: How Many, Model with Numbers, Shapes in the World, Making Shapes from Parts, Bigger or Equal, Place and Position of Numbers

Figure 6.8a. Grade K Content Connections, Big Ideas, and Standards

Content Connection	Big Idea	K Standards
Communicating Stories with Data	Sort & Describe Data	MD.1, MD.2, MD.3, CC.4, CC.5, G.4: Sort, count, classify, compare, and describe objects using numbers for length, weight, or other attributes.
Exploring Changing Quantities	How Many?	CC.1, CC.2, CC.3, CC.4, CC.5, CC.6, CC.7, MD.3: Know number names and the count sequence to determine how many are in a group of objects arranged in a line, array, or circle. Fingers are important representations of numbers. Use words and drawings to make convincing arguments to justify work.
Exploring Changing Quantities	Bigger or Equal?	CC.4, CC.5, CC.6, MD.2, G.4: Identify a number of objects as greater than, less than, or equal to the number of objects in another group. Justify or prove your findings with number sentences and other representations.
Taking Wholes Apart, Putting Parts Together	Being Flexible within 10	OA.1, OA.2, OA.3, OA.4, OA.5, CC.6, G.6: Make 10, add and subtract within 10, compose and decompose within 10 (find 2 numbers to make 10). Fingers are important.
Taking Wholes Apart, Putting Parts Together	Place and position of numbers	CC.3, CC.5, NBT.1: Get to know numbers between 11 and 19 by name and expanded notation to become familiar with place value, for example: $14 = 10 + 4$.

Content Connection	Big Idea	K Standards
Taking Wholes Apart, Putting Parts Together	Model with numbers	OA.1, OA.2, OA.5, NBT.1, MD.2: Add, subtract, and model abstract problems with fingers, other manipulatives, sounds, movement, words, and models.
Discovering Shape and Space	Shapes in the World	G.1, G.2, G.3, G.4, G.5, G.6, MD.1, MD.2, MD.3: Describe the physical world using shapes. Create 2-D and 3-D shapes, and analyze and compare them.
Discovering Shape and Space	Making shapes from parts	MD.1, MD.2, G.4, G.5, G.6: Compose larger shapes by combining known shapes. Explore similarities and differences of shapes using numbers and measurements.

Grade One

Organizing and seeing equivalence are ideas that pervade first grade. Students develop ways to organize to help them with counting and comparing and ultimately understanding the place value system. Grade 1 students will compare two, two-digit numbers based on the meanings of the tens and the ones digits, which is an important concept (SMP.1, 2; 1.NBT.3). To gain this understanding, students have worked extensively creating tens from collections of ones and have internalized the idea of a “ten.” Younger learners typically count by ones, and may show little or no grouping or organization of objects as they count. As they acquire greater confidence and understanding, children can progress to counting some of the objects in groups of five or ten. Teachers may support student learning by providing interesting, varied, and frequent counting opportunities using games, group activities, and a variety of tools, along with focused mathematical discourse. Students who are English learners are encouraged to use their developing English and native language assets and draw on their prior knowledge. Teachers can provide purposefully planned and “just-in-time” scaffolds and supports to engage EL students in sustained mathematical oral discourse in multiple contexts to build academic vocabulary and knowledge.

Equivalence means learning to assess what makes things different and the same. For instance, $4 + 1$ and 5 are equivalent, even though they look different, and students may develop a dozen strategies for adding 4 and 1 to arrive at 5 . Those strategies are different but related and equivalent in the result they produce. Grappling with equivalence and organization is important work in first grade.

Posing questions as students are engaged in the activities can help a child see relationships and further develop place value concepts. Some questions might include the following:

- What do you notice?
- What do you wonder?

- What will happen if we count these by ones?
- What if we counted them in groups of ten?
- How can we be sure there really are 43 here?
- I see you counted by groups of 10 and ones. What if you counted them all by ones? How many would we get?

Teachers can have students assemble bundles of ten objects (popsicle sticks or straws, for example), or snap together linking cubes to make tens as a means of developing the concept and noting how the quantities are related. Note that while students in first grade do begin to add two-digit numbers, they do so using *strategies* as distinguished from formal *algorithms*. The California Common Core State Standards for Mathematics (CA CCSSM) intentionally place the introduction of a standard algorithm for addition and subtraction in fourth grade (4.NBT.4). Examples of useful manipulatives at this age include 10-frames, Rekenreks, comparison bars, Cuisenaire rods, and useful visuals include hundreds charts, 0–99 charts, and number paths. Fingers continue to be important. NRICH provides online Cuisenaire Rods, and other moveable shapes.⁶

In first grade, students can conduct data investigations, generating questions to study, using measurements of length and time, along with continued work categorizing and counting objects, and categorizing geometric objects by attributes. When conducting data investigations, it is important to avoid questions about students' physical attributes or possessions, even those that seem innocuous, such as height or arm length. Instead, some good questions to wonder about might be "I wonder what time it will be when the next person walks into the classroom?" or "I wonder which book in the classroom is the most read?," comparing events or objects rather than personal characteristics. Guidance cards can provide additional support to help EL students engage in structured explorations of the big ideas (What you can do) and communicate (What you can say) with peers.

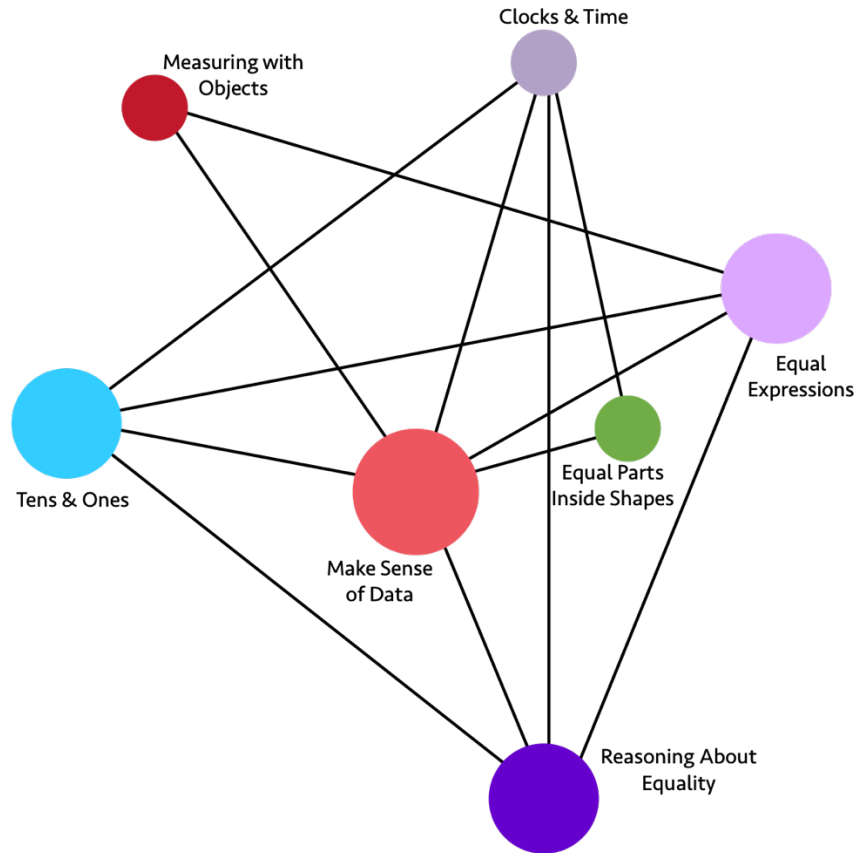
Students extend their work from kindergarten, focusing on two dimensional shapes in a flat surface, to considering ways these shapes are the faces of 3-D shapes that make up the world. Students can work qualitatively and quantitatively with shapes, using their language to describe the similarities and differences, and counting and joining numbers to describe the shapes. For example, a student might notice a cube has 4 corners when looking directly at the square that forms one of its 6 faces. Students can count the corners, or vertices, and notice that a cube has 8 vertices and 6 square faces. A student may then notice that a prism has the same number of faces and vertices, but four of the faces are rectangles and the other two faces are squares. Including the circle as an additional shape brings in discussion about cylinders and cones. The circle is an important shape to discuss, as all circles are similar. Students also see the circle as an item that can be constructed from sectors or pieces. Constructing circles, and playing with

⁶ <https://nrich.maths.org/>

pieces that combine to make a circle, begins an important journey towards fractions and telling the time on an analog clock.

Critical Areas of Instructional Focus

Figure 6.9. Grade 1 Big Ideas



Long description: The graphic illustrates the connections and relationships of some first-grade mathematics concepts. Direct connections include:

- Clocks & Time directly connects to: Equal Parts Inside Shapes, Reasoning About Equality, Make Sense of Data, Tens & Ones
- Equal Expressions directly connects to: Reasoning About Equality, Make Sense of Data, Tens & Ones, Measuring with Objects
- Reasoning About Equality directly connects to: Equal Expressions, Clocks & Time, Make Sense of Data, Tens & Ones

- Tens & Ones directly connects to: Reasoning About Equality, Make Sense of Data, Equal Expressions, Clocks & Time
- Measuring with Objects directly connects to: Equal Expressions, Make Sense of Data
- Equal Parts Inside Shapes directly connects to: Clocks & Time, Make Sense of Data
- Make Sense of Data directly connects to: Reasoning About Equality, Tens & Ones, Measuring with Objects, Clocks & Time, Equal Expressions, Equal Parts Inside Shapes

Figure 6.9a. Grade 1 Content Connections, Big Ideas, and Standards

Content Connection	Big Idea	Grade 1 Standards
Communicating Stories with Data	Make Sense of Data	MD.2, MD.4, MD.3, MD.1, NBT.1, OA.1, OA.2, OA.3: Organize, order, represent, and interpret data with two or more categories; ask and answer questions about the total number of data points, how many are in each category, and how many more or less are in one category than in another.
Communicating Stories with Data & Exploring Changing Quantities	Measuring with Objects	MD.1 MD.2, OA.5: Express the length of an object by units of measurement e.g., the stapler is 5 red Cuisenaire rods long, the red rod representing the unit of measure. Understand that the measurement length of an object is the number of units used to measure.
Exploring Changing Quantities	Clocks & Time	MD.3, NBT.2, G.3: Read and express time on digital and analog clocks using units of an hour or half hour.
Exploring Changing Quantities	Equal Expressions	OA.6, OA.7, OA.2, OA.1, OA.8, OA.5, OA.4, OA.3, NBT.4: Understand addition and subtraction, using various models, such as connected cubes. Compose and decompose numbers to make equal expressions, knowing that equals means that both sides of an expression are the same (and it is not simply the result of an operation).
Exploring Changing Quantities	Reasoning about Equality	OA.3, OA.6, OA.7, NBT.2, NBT.3, NBT.4: Justify reasoning about equal amounts, using flexible number strategies (e.g., students use compensation strategies to justify number sentences, such as $23 - 7 = 24 - 8$).

Content Connection	Big Idea	Grade 1 Standards
Taking Wholes Apart, Putting Parts Together	Tens & Ones	NBT.4, NBT.3, NBT.1, NBT.2, NBT.6, NBT.5: Think of whole numbers between 10 and 100 in terms of tens and ones. Through activities that build number sense, students understand the order of the counting numbers and their relative magnitudes.
Discovering Shape and Space	Equal Parts inside Shapes	G.3, G.2, G.1, MD.3: Compose 2D shapes on a plane as well as in 3D space to create cubes, prisms, cylinders, and cones. Shapes can also be decomposed into equal shares, as in a circle broken into halves and quarters defines a clock face.

Grade Two

In second grade students start to think deeply about familiar benchmark or “friendly” numbers, so they can use them to compose, decompose, and compare numbers. In second grade, students extend their understanding of place value and number comparison to include three-digit numbers. To compare two three-digit numbers, second graders can take the number apart by place value and compare the number of hundreds, tens, and ones, or they may use counting strategies. Thinking with numbers, such as ones, tens, and hundreds, and negotiating how to use them as groups and as positions on the number line to solve problems, is central to this grade. Students continually anchor their thinking about number to all the real-world places where numbers are used to describe and wonder, including estimating lengths and quantities and thinking with data. Note that while students in second grade do begin to subtract numbers, they do so using *strategies* as distinguished from formal *algorithms*.

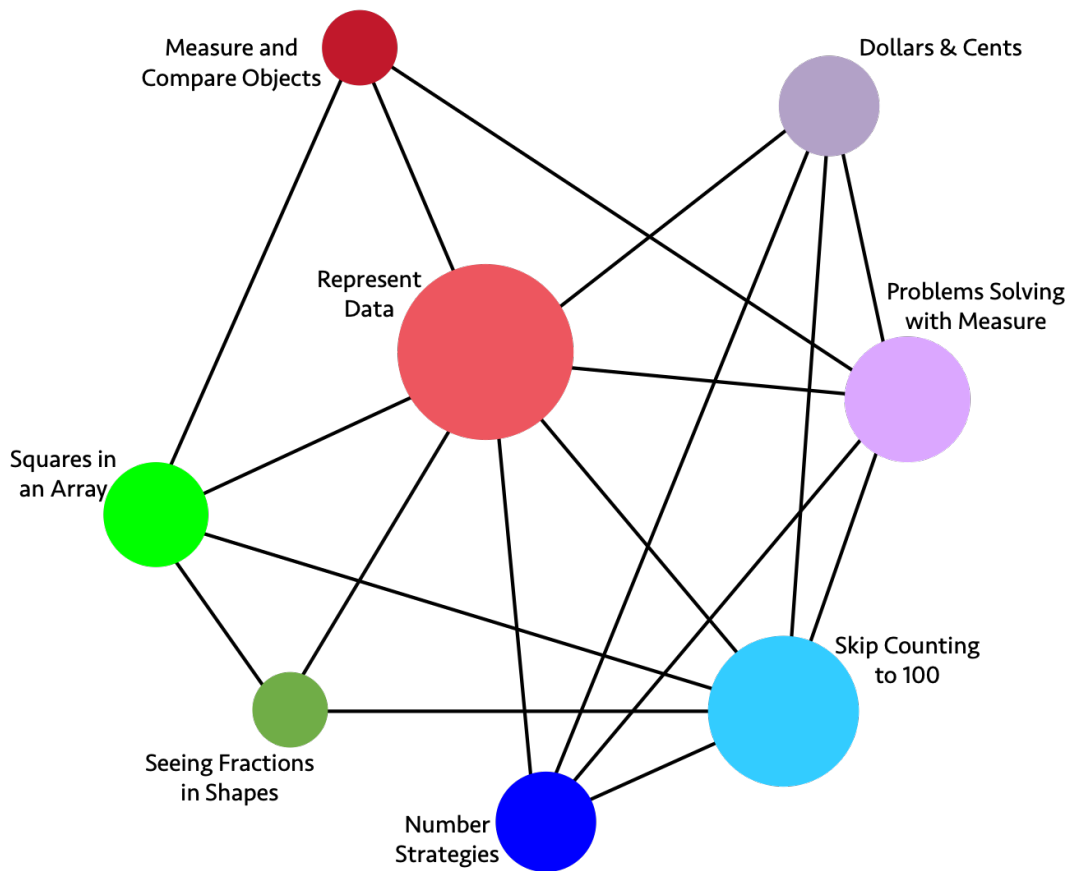
In second grade, students can conduct data investigations and interpret data visuals through data talks. Students continue to use measurement of length and time as contexts in generating questions, along with continued work categorizing and counting objects and categorizing geometric objects by attributes. In second grade, students also start to use the context of money. When conducting data investigations, it is important to avoid questions about students’ physical attributes or possessions, even those that seem innocuous, such as hair color or shoe type. Instead, some good questions to wonder about might be “I wonder what time it will be when the next person walks into the classroom?” or “I wonder which book in the classroom is the most read?,” comparing events or objects rather than personal characteristics. Students who are English learners are encouraged to use their developing English and native language assets and draw on their prior knowledge. Teachers can examine text and tasks for key language forms and structures, and general as well as discipline-specific, high-utility academic vocabulary words linked to the big ideas and connections. Teachers can provide purposefully planned and “just-in-time” scaffolds and supports to engage EL students in sustained mathematical discourse in multiple contexts to build academic language and knowledge.

Grade 2 begins more formal vocabulary use and the connections between numbers and shapes. Students continue to work with 2-D and 3-D shapes, composing and decomposing within a plane or space, as they refine their understanding of figures that have area and those that have volume. Students begin to partition shapes into equal units, known as unit fractions. This is especially important in circles and will be the base understanding of navigating time using an analog clock.

Students learn in second grade that they can partition rectangles into arrays of equal squares and quantify the lengths of sides by using a unit to measure. For example, a student may use a light green Cuisenaire rod to approximate the length of a stapler or a stack of 5 Unifix cubes to measure the same length. This can lead to discussion of the importance of a base unit as the length that is used to quantify. Equal partitions are also of utmost importance, as students begin to understand the idea of a fraction.

Critical Areas of Instructional Focus

Figure 6.10. Grade 2 Big Ideas



Long description: The graphic illustrates the connections and relationships of some second-grade mathematics concepts. Direct connections include:

- Dollars & Cents directly connects to: Problems Solving with Measure, Skip Counting to 100, Number Strategies, Represent Data
- Problems Solving with Measure directly connects to: Skip Counting to 100, Number Strategies, Represent Data, Measure and Compare Objects, Dollars & Cents
- Skip Counting to 100 directly connects to: Number Strategies, Seeing Fractions in Shapes, Squares in an Array, Represent Data, Dollars & Cents, Problems Solving with Measure
- Number Strategies directly connects to: Skip Counting to 100, Problems Solving with Measure, Dollars & Cents, Represent Data

- Seeing Fractions in Shapes directly connects to: Skip Counting to 100, Represent Data, Squares in an Array
- Squares in an Array directly connects to: Seeing Fractions in Shapes, Skip Counting to 100, Represent Data, Measure and Compare Objects
- Measure and Compare Objects directly connects to: Squares in an Array, Represent Data, Problems Solving with Measure
- Represent Data directly connects to: Measure and Compare Objects, Dollar & Cents, Problems Solving with Measure, Skip Counting to 100, Number Strategies, Seeing Fractions in Shapes, Squares in an Array

Figure 6.10a. Grade 2 Content Connections, Big Ideas, and Standards

Content Connection	Big Idea	Standards
Communicating Stories with Data	Measure & Compare Objects	MD.1, MD.2, MD.3, MD.4, MD.6, MD.9: Determine the length of objects using standard units of measures, and use appropriate tools to classify objects, interpreting and comparing linear measures on a number line.
Communicating Stories with Data	Represent Data	MD.7, MD.9, MD.10, G.2, G.3, NBT.2: Represent data by using line plots, picture graphs, and bar graphs, and interpret data in different data representations, including clock faces to the nearest 5 minutes.
Exploring Changing Quantities	Dollars & Cents	MD.8, MD.5, NBT.1, NBT.2, NBT.5, NBT.6, NBT.7: Understand the unit values of money and compute different values when combining dollars and cents.
Exploring Changing Quantities & Discovering Shape and Space	Problem Solving with Measure	NBT.7, NBT.1, MD.1, MD.2, MD.3, MD.4, MD.5, MD.6, MD.9, OA.1: Solve problems involving length measures using addition and subtraction.
Taking Wholes Apart, Putting Parts Together	Skip Counting to 100	NBT.1, NBT.3, NBT.7, OA.4, G.2: Use skip counting, counting bundles of 10, and expanded notation to understand the composition and place value of numbers up to 1,000. This includes ideas of counting in fives, tens, and multiples of hundreds, tens, and ones, as well as number relationships involving these units, including comparing.

Content Connection	Big Idea	Standards
Taking Wholes Apart, Putting Parts Together	Number Strategies	MD.5, NBT.5, NBT.6, NBT.7, OA.1, OA.2: Add and subtract 2-digit numbers, within 100, without using algorithms - instead encouraging different strategies and justification. Compare and contrast the different strategies using models, symbols, and drawings.
Discovering Shape and Space	Seeing Fractions in Shapes	G.1, G.2, G.3, MD.7: Divide circles and rectangles into equal shares and know them to be standard unit fractions. Identify and draw 2D and 3D shapes, recognizing faces and angles.
Discovering Shape and Space	Squares in an Array	OA.4, G.2, G.3, MD.6: Partition rectangles into rows and columns of unit squares to find the total number of square units in an array.

Chapter 7: Mathematics in Grades Three through Five

Figure 7.1. A Progression Chart of Big Ideas through Grades 3–5

Content Connections	Big Ideas: Grade 3	Big Ideas: Grade 4	Big Ideas: Grade 5
Communicating Stories with Data	Represent Multivariable data	Measuring and plotting	Plotting patterns
Communicating Stories with Data	Fractions of shape and time	Rectangle Investigations	Telling a data story
Communicating Stories with Data	Measuring	n/a	n/a
Exploring Changing Quantities	Addition and subtraction patterns	Number and shape patterns	Telling a data story
Exploring Changing Quantities	Number flexibility to 100	Factors & area models	Factors and groups
Exploring Changing Quantities	n/a	Multi-digit numbers	Modeling
Exploring Changing Quantities	n/a	n/a	Fraction connections
Exploring Changing Quantities	n/a	n/a	Shapes on a plane
Taking Wholes Apart, Putting Parts Together	Square tiles	Fraction flexibility	Fraction connections
Taking Wholes Apart, Putting Parts Together	Fractions as relationships	Visual fraction models	Seeing Division
Taking Wholes Apart, Putting Parts Together	Unit fraction models	Circles, fractions and decimals	Powers and place value
Discovering shape and space	Unit fraction models	Circles, fractions and decimals	Telling a data story
Discovering shape and space	Analyze quadrilaterals	Shapes and symmetries	Layers of cubes
Discovering shape and space	n/a	Connected problem solving	Shapes on a plane

The upper-elementary grades present new opportunities for developing and extending number sense. There are four big ideas related to number sense for grades 3–5 including:

- extending flexibility with numbers,

- understanding the operations of multiplication and division,
- making sense of operations with fractions and decimals, and
- using number lines as tools.

As students learn to think about numbers flexibly, by composing and decomposing numbers, they will learn to recognize the inverse relationship between addition and subtraction and between multiplication and division. An important component of “fluency” is being flexible with numbers; rather than a focus on being fast with computation or the use of damaging timed tests (Boaler, 2019). If students are given meaningful explorations with numbers and number patterns, they will develop memories of mathematics facts, and the memories will be meaningful and conceptual. As students learn in these grades to identify and express patterns, both visually and numerically, they will build foundations for proportional reasoning when thinking about the connections between units. In fifth grade, the flexibility students have developed with numbers can be applied to fractions and to the place value system.

Students in the upper-elementary grades learn to conduct data investigations, which include asking and answering questions that are of interest to them. They learn to collect and analyze data, determine, and confirm results, and communicate their findings. While the data visualizations set out in the standards in these grades only include picture graphs, bar graphs, and line plots, students do not need to be restricted to these.

Students investigate patterns and relationships in two- and three-dimensional space, and they begin to use the coordinate plane to represent and question relationships. As students learn about three-dimensional space, they build understanding of the volume as a quantity of unit cubes that fill the space of a solid. Students study time as a measure and connect the central angles of a circle to the clock face and hands. The study of the clock face is another area for connections between numbers and fractions, as students learn about and communicate detailed measures of time.

Students who are English learners are encouraged to use their developing English and native language assets and draw on their prior knowledge. Teachers can examine text and tasks for key language forms and structures, and general as well as discipline-specific, high-utility academic vocabulary words linked to the big ideas and connections. Teachers can provide purposefully planned and “just-in-time” scaffolds and supports to engage EL students in sustained mathematical discourse in multiple contexts to build academic language and knowledge.

The following vignette highlights an educator who is using digital tools to allow students to better visualize concepts introduced at this grade span.

Vignette: Pace, Flexibility, and Grace—Teaching Math During Times of Change

At the end of the 2015-16 school year, 32-year veteran math teacher Jean Maddox participated in a three-year professional development program with the County of Tulare Office of

Education. The Central Valley Network Improvement Community program or CVNIC was, as the Visalia Unified teacher reflects, “The best professional development I have ever participated in.” Jean, who has taught 5th grade for the last 20 years, worked with other teachers in the program and mathematics researchers, Jo Boaler and Graham Fletcher, to learn new math strategies while applying their knowledge using the PDSA cycle (Plan, Do, Study, Act). Little did she know then just how useful those skills would become in 2020.

Forced by the pandemic to teach remotely in 2020, Jean rose to the challenge by adapting, collaborating, and innovating. Even though much had changed with respect to how Jean connected with her students this past year, the more important stuff—the motivation behind her work, the aspirations that motivate her students, even the personal character of her outreach—remained.

Flexibility with numbers is a crucial aspect of Jean’s work with her students, making sure they are confident and comfortable with the way they see and manipulate numbers. Often, Jean encourages students to explain the problem they are working on in two ways, with numbers and then visually. “How they see it, in my opinion, also shows how flexible they are with numbers,” Jean says. “This especially helps when we are solving math stories with fractions. How the story makes sense in their head, to match it up with a visual they have drawn; finally, their thinking allows me to see their true understanding (or misconceptions) of the problem.” In the distance learning setting, Jean works with tasks from YouCubed that are a “low-floor/high-ceiling” and fit well for 5th graders beginning to look for patterns to explain their thinking—both visually and numerically.

Using flexibility allows Jean’s students to develop the idea of efficiency when doing long division, a key concept developed at this grade span. “For a lot of my students, using the area model, whether it be for multiplication or division, allows them the chance for a visual/conceptual image with numbers, especially division,” Jean states. For long division, the area model is a way to organize the method of Partial Quotients, which allows Jean’s students to work with numbers they are comfortable with when they are dividing. The model enables them to connect to the understanding of areas that they learned in third grade.

Jean’s teaching method engages all learners. For EL students, Jean has found success through math stories (word problems). “Students draw out what they understand first (because visualizing the math before calculation shows true understanding),” says Jean. “I provide a process where students can share their thinking, whether it be by partner discussions, acting out the problem, explaining their drawings, and then finally doing the number work (calculations) is the last step.” The process allows Jean to drop in on her students’ thinking process, which indicates to her what the next steps are for working with each student.

Math manipulatives to represent concepts introduced at this grade span are central in Jean’s students’ work. “Anything from fraction strips, tangrams, Unifix cubes, centimeter cubes, rulers, protractors are used daily; the students can use the manipulatives they are the most comfortable with,” says Jean. It is a norm in her classroom that students may choose the appropriate tools that they need during math. Jean innovated to ensure the pandemic didn’t

change the norm. “With technology, during our full-distance learning time, students have access to digital manipulatives because they still need to be able to see and create the math they are being asked to conceptualize.” Jean found the solution in the Math Learning Center website⁷ and the Toy Theater website⁸ to access manipulatives for her students’ use. “They have been quite successful in sharing their thinking, so I can see what their misconceptions are and what they understand (formative assessments).”

On the first day of math instruction, Jean makes sure students have a clear understanding of whole numbers on a number line, allowing them to make a smoother transition to placing fractions on a number line. Jean found the Math Learning Center website to be a valuable resource for using digital number lines.

For foundational algebra skills, Jean uses Jamboard [collaborative digital whiteboard] to have students manipulate equations and SolveMe Mobiles [game-based mathematics website] to help students establish algebraic thinking and reinforce Operations and Algebraic standards from previous grades.

Pace is another critical factor in Jean’s teaching. In her classes, whether online or in-person, speed is not an indicator of success. “I strongly disagree with timed tests,” she says. “It only heightens a student's math anxiety. Watching the anxious look on kids' faces with the knowledge that they may not finish, which then leads them to feel like a failure and that they are not a math person.” For Jean, timed-tests give students the impression that math is about speed and getting the correct answer - the very opposite of what math is. “In my class, we take time as we are doing math because I want students to think deeply and realize that their math ability does not connect to how fast they can get something done.”

Success for Jean is the lightbulb moment when a student connects with the work. “When a student can take a number and decompose it so they can add or multiply in a way that makes sense to them, one that is not necessarily the standard algorithm. When they get the correct answer, they look at the teacher and say, ‘This is how I see it!’ That is an amazing experience—especially as their teacher.”

Grade Three

In third grade, students extend their work from second grade, thinking with groups, to equal groups and rows and columns in multiplication. As students learn to think about numbers flexibly, by composing and decomposing numbers, they will learn to recognize the inverse relationship between multiplication and division. Being flexible with numbers is an important component of fluency, rather than a focus on being fast with computation. The 2021 *Mathematics Framework* defines fluency:

⁷ <https://www.mathlearningcenter.org/apps>

⁸ <https://toytheater.com/category/teacher-tools/virtual-manipulatives/>

Students who are comfortable with numbers and who have learned to compose and decompose numbers strategically develop fluency along with conceptual understanding. In the past, fluency has sometimes been equated with speed, which may account for the common, but counterproductive, use of timed tests for practicing facts. But in fact, research has found that, “Timed tests offer little insight about how flexible students are in their use of strategies or even which strategies a student selects. And evidence suggests that efficiency and accuracy may actually be negatively influenced by timed testing.”

To learn more about why timed tests can be replaced with number sense activities see Boaler, Williams, and Confey (2015).⁹ Being flexible is a big idea and one which draws from connections between numbers and patterns. When students develop number sense, they have a flexible internal framework that they can draw upon when working with any mathematics. Note that while students in third grade do begin to divide numbers, they do so using *strategies* as distinguished from formal *algorithms*.

Third grade is also the time when fractional thinking begins to become robust and can begin with a deep understanding of one-half that students can build on to understand and visualize other unit fractions.

Students can be given plenty of time to “play” with numbers and fractions, to think about their relative size, and to estimate and reflect on whether their answers make sense (SMP.3, 7, 8). Students in third grade focus on understanding fractions as equal parts of a whole and as numbers located on the number line. They also use reasoning to compare unit fractions (3.NF.1, 2, 3).

In third grade, students can conduct data investigations and interpret data visuals through data talks. Contexts for questions to investigate should expand to include volume and mass measurement (grams, kilograms, and liters, but not compound units, such as cm^3) in addition to the length, time, and money contexts from earlier grades (3.MD.A.2). Time measurements are refined to the nearest minute (3.MD.A.1), and length now includes half- and quarter-inches (3.MD.B.4). Beginning ideas of area give another possible context, limited here to areas that can be covered by a whole number of unit squares (3.MD.C.5, 3.MD.C.6).

Students continue refining their understanding of two-dimensional shapes, focusing on the similarities and differences between quadrilaterals. Students make sense of the attributes that make up these important shapes, that can all be composed by triangles. Students will recognize the importance of the triangle as the base unit, which connects with later learning of trigonometry and the method of dividing complex shapes into smaller triangles to find their area. Students investigate and quantify quadrilaterals and learn that area and perimeter are important measures, where perimeter is one dimensional and area is two dimensional. Having conversations about the base unit of measure helps students connect to the ideas of

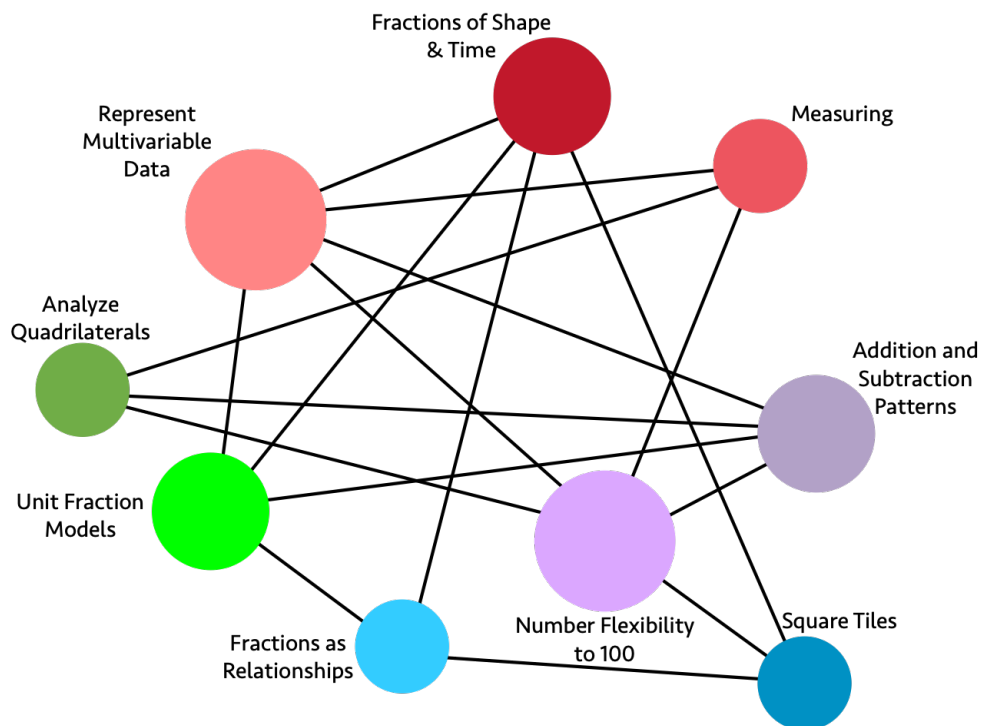
⁹ <https://www.youcubed.org/evidence/fluency-without-fear/>

multiplication and division. Connecting numbers and shapes helps students to make other connections and build number sense.

Students who are English learners are encouraged to use their developing English and native language assets (e.g., cognates) and draw on their prior knowledge. Teachers can examine text and tasks for key language forms and structures, in addition to general and discipline-specific, high-utility academic vocabulary words linked to the big ideas and connections. Teachers can provide purposeful scaffolds and supports to engage EL students in sustained mathematical discourse in multiple contexts to build academic language and knowledge. Planned and “just-in-time” scaffolds and supports provide multiple entry points for meaning-making and sharing of ideas in mathematical ways and include representations, expression starters and builders, and targeted vocabulary and language structures (e.g., explanation, descriptions, comparisons, methods, and connections).

Critical Areas of Instructional Focus

Figure 7.2. Grade 3 Big Ideas



Long description: The graphic illustrates the connections and relationships of some third-grade mathematics concepts. Direct connections include:

- Fractions of Shape & Time directly connects to: Square Tiles, Fractions as Relationships, Unit Fraction Models, Represent Multivariable Data
- Measuring directly connects to: Number Flexibility to 100, Analyze Quadrilaterals, Represent Multivariable Data
- Addition and Subtraction Patterns directly connects to: Number Flexibility to 100, Unit Fraction Models, Analyze Quadrilaterals, Represent Multivariable Data
- Square Tiles directly connects to: Fractions as Relationships, Number Flexibility to 100, Fractions of Shape & Time
- Fractions as Relationships directly connects to: Square Tiles, Fractions of Shape & Time, Unit Fraction Models
- Unit Fraction Models directly connects to: Fractions as Relationships, Addition and Subtraction Patterns, Fractions of Shape & Time, Represent Multivariable Data
- Analyze Quadrilaterals directly connects to: Number Flexibility to 100, Addition and Subtraction Patterns, Measuring
- Represent Multivariable Data directly connects to: Unit Fraction Models, Number Flexibility to 100, Addition and Subtraction Patterns, Measuring, Fractions of Shape & Time
- Number Flexibility to 100 directly connects to: Square Tiles, Analyze Quadrilaterals, Represent Multivariable Data, Measuring, Addition and Subtraction Patterns

Figure 7.2a. Grade 3 Content Connections, Big Ideas, and Standards

Content Connection	Big Idea	Grade 3 Standards
Communicating Stories with Data	Represent Multivariable Data	MD.3, MD.4, MD.1, MD.2, NBT.1: Collect data and organize data sets, including measurement data; read and create bar graphs and pictographs to scale. Consider data sets that include three or more categories (multivariable data) for example, when I interact with my puppy, I either call her name, pet her, or give her a treat.

Content Connection	Big Idea	Grade 3 Standards
Communicating Stories with Data & Taking Wholes Apart, Putting Parts Together & Discovering Shape and Space	Fractions of Shape & Time	MD.1, NF.1, NF.2, NF.3, G.2: Collect data by time of day, show time using a data visualization. Think about fractions of time and of shape and space, expressing the base unit as a unit fraction of the whole.
Communicating Stories with Data	Measuring	MD.2, MD.4, NBT.1: Measure volume and mass, incorporating linear measures to draw and represent objects in two-dimensional space. Compare the measured objects, using line plots to display measurement data. Use rounding where appropriate.
Exploring Changing Quantities	Addition and Subtraction Patterns	NBT.2, , OA.8, OA.9, MD.1: Add and subtract within 1000 - Using student generated strategies and models, such as base 10 blocks. e.g., use expanded notation to illustrate place value and justify results. Investigate patterns in addition and multiplication tables, and use operations and color coding to generalize and justify findings.
Exploring Changing Quantities	Number Flexibility to 100	OA.1, OA.2, OA.3, OA.4, OA.5, OA.6, OA.7, OA.8, NBT.3, MD.7, NBT.1: Multiply and divide within 100 and justify answers using arrays and student generated visual representations. Encourage number sense and number flexibility - not “blind” memorization of number facts. Use estimation and rounding in number problems.
Taking Wholes Apart, Putting Parts Together	Square Tiles	MD.5, MD.6, MD.7, OA.7, NF.1: Use square tiles to measure the area of shapes, finding an area of n squared units, and learn that one square represents 1/nth of the total area.
Taking Wholes Apart, Putting Parts Together	Fractions as Relationships	NF.1, NF.3: Know that a fraction is a relationship between numerators and denominators – and it is important to consider the relationship in context. Understand why $1/2=2/4=3/6$.

Content Connection	Big Idea	Grade 3 Standards
Discovering Shape and Space & Taking Wholes Apart, Putting Parts Together	Unit Fraction Models	NF.2, NF.3, MD.1: Compare unit fractions using different visual models including linear models (e.g., number lines, tape measures, time, and clocks) and area models (e.g., shape diagrams encourage student justification with visual models).
Discovering Shape and Space	Analyze Quadrilaterals	MD.8, G.1, G.2, NBT.1, OA.8: Describe, analyze, and compare quadrilaterals. Explore the ways that area and perimeter change as side lengths change, by modeling real world problems. Use rounding strategies to approximate lengths where appropriate.

Grade Four

Patterning and examining relationships are at the heart of fourth grade. Students begin to think about how to identify and express patterns, both visually and numerically, and build foundations for proportional reasoning when thinking about the connections between units. Students look within fractions and decimals for the relationships represented there—relationships between numerator and denominator, fraction and decimal, and decimal and place value. Fourth graders use relationships to connect multiplication and division and think flexibly across all operations.

After their introduction to multiplication in third grade, fourth-grade students employ that understanding to identify prime and composite numbers and to recognize that a whole number is a multiple of each of its factors. An excellent way for students to see the composition of numbers is the visual number activity.¹⁰ Students can also explore the multiplication table and highlight multiples with color or shape, looking for patterns and relationships

At this grade, students develop an understanding of fraction equivalence by illustrating and explaining reasons for their conjectures and ideas. Students can strengthen their knowledge of fraction equivalence by engaging in games that provide practice, such as Matching Fractions or Fractional Wall, created by Nrich Maths.¹¹ Students represent their thinking with diagrams (number lines, strip diagrams), pictures, and equations. This work lays the foundation for further operations with fractions in fifth grade.

Data investigations in fourth grade should include topics of student interest as students learn the ways to collect, analyze, and represent data. Line plots are introduced in fourth grade, and

¹⁰ <https://www.youcubed.org/wim/number-visuals-3-5/>

¹¹ <https://nrich.maths.org/>

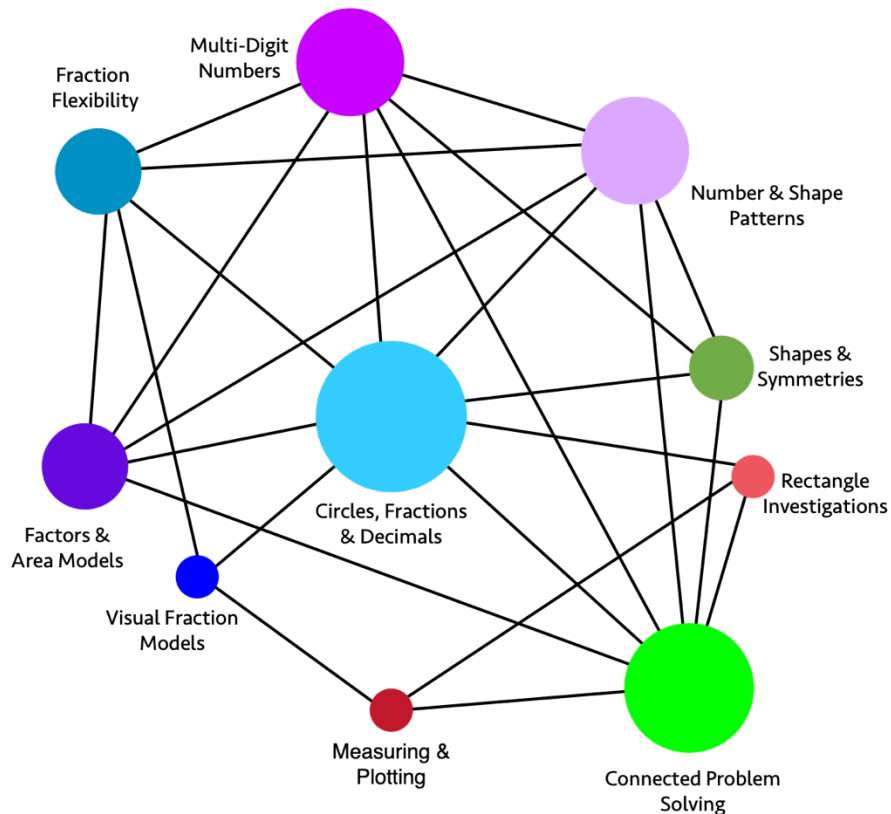
students can learn to create, read, and interpret different data displays, including line plots. When creating line plots, students can include fractional measurements to help bring fractions to life with real data, such as measurement of objects in the classroom or home.

In grade four, students move from seeing vertices as made up of an angle to more formal understandings of angles made of two rays with a common endpoint. The concept of a ray can lead to fascinating discussions of infinity that can captivate students. As students think about the addition of angles, they will again be connecting geometric ideas to number sense. The idea of a central angle of a circle, formed when two rays are joined at the center of a circle can connect with learning about the hands of a clock face. Students can investigate with angles in myriad of ways. Students continue refining their work in measuring and quantifying the world around them by investigations, such as connecting the unit of measure from a square to a cube, as they use area, perimeter of shapes, and the volume of solids. These ideas connect to the operations of addition, subtraction, multiplication, and division. Students connect the unit of measure from a square to a cube as they use area and volume to make sense of space. These ideas can also be connected to fractions and decimals, building visual understandings and helping with the meaning of these operations.

Students who are English learners are encouraged to use their developing English and native language assets (e.g., cognates, morphological awareness) and draw on their prior knowledge. Teachers can examine text and tasks for key language forms and structures, and general as well as discipline-specific, high-utility academic vocabulary words linked to the big ideas and connections. Teachers can provide purposeful scaffolds and supports to engage EL students in sustained mathematical discourse in multiple contexts to build academic language and knowledge. Planned and “just-in-time” scaffolds and supports provide multiple entry points for meaning making and sharing of ideas in mathematical ways and include representations, expression starters and builders, and targeted academic vocabulary and language structures (e.g., problems, explanations, arguments, descriptions, procedures). Teachers guide deconstruction and/or co-construction of problems, investigations, arguments, explanations, descriptions, and procedures.

Critical Areas of Instructional Focus

Figure 7.3. Grade 4 Big Ideas



Long description: The graphic illustrates the connections and relationships of some fourth-grade mathematics concepts. Direct connections include:

- Number & Shape Patterns directly connects to: Shapes & Symmetries, Connected Problem Solving, Circles Fractions & Decimals, Factors & Area Models, Fraction Flexibility, Multi-Digit Numbers
- Shapes & Symmetries directly connects to: Connected Problem Solving, Circles Fractions & Decimals, Multi-Digit Numbers, Number & Shape Patterns
- Rectangle Investigations directly connects to: Connected Problem Solving, Measuring & Plotting, Circles Fractions & Decimals
- Connected Problem Solving directly connects to: Rectangle Investigations, Shapes & Symmetries, Number & Shapes Patterns, Multi-Digit Numbers, Circles Fractions & Decimals, Factors & Area Models, Measuring & Plotting
- Measuring & Plotting directly connects to: Connected Problem Solving, Rectangle Investigations, Visual Fraction Models

- Visual Fraction Models directly connects to: Measuring & Plotting, Circles Fractions & Decimals, Fraction Flexibility
- Factors & Area Models directly connects to: Connected Problem Solving, Circles Fractions & Decimals, Number & Shape Patterns, Multi-Digit Numbers, Fraction Flexibility
- Fraction Flexibility directly connects to: Factors & Area Models, Circles Fractions & Decimals, Number & Shape Patterns, Multi-Digit Numbers
- Multi-Digit Numbers directly connects to: Number & Shape Patterns, Shapes & Symmetries, Connected Problem Solving, Circles Fractions & Decimals, Factors & Area Models, Fraction Flexibility
- Circles Fractions & Decimals directly connects to: Multi-Digit Numbers, Number & Shape Patterns, Shapes & Symmetries, Rectangle Investigations, Connected Problem Solving, Visual Fraction Models, Factors & Area Models, Fraction Flexibility

Figure 7.3a. Grade 4 Content Connections, Big Ideas, and Standards

Content Connection	Big Idea	Grade 4 Standards
Communicating Stories with Data	Measuring & Plotting	MD.1, MD.4, NF.1, NF.2: Collect data consisting of distance, intervals of time, volume, mass, or money. Read, interpret, and create line plots that communicate data stories where the line plot measurements consist of fractional units of measure. For example, create a line plot showing classroom or home objects measured to the nearest quarter inch.
Communicating Stories with Data	Rectangle Investigations	MD1, MD2, MD3, MD5, MD6: Investigate rectangles in the world, measuring lengths and angles, collecting the data, and displaying it using data visualizations.
Exploring Changing Quantities	Number & Shape Patterns	OA.5, OA.1, OA.2, NBT.4: Generalize number and shape patterns that follow a given rule. Communicate understanding of how the pattern changes in words, symbols, and diagrams - working with multi-digit numbers.
Exploring Changing Quantities	Factors & Area Models	OA.1, OA.2, OA.4, NBT.5, NBT.6: Break numbers inside of 100 into factors. Illustrate whole number multiplication and division calculations as area models and rectangular arrays that illustrate factors.
Exploring Changing Quantities	Multi-Digit Numbers	NBT.1, NBT.2, NBT 3, NBT.4, OA.1: Read and write multi-digit whole numbers in expanded form and express each number component of the expanded form as a multiple of a power of ten.

Content Connection	Big Idea	Grade 4 Standards
Taking Wholes Apart, Putting Parts Together	Fraction Flexibility	NF.3, NF.1, NF.4, NF.5, OA.1: Understand that addition and subtraction of fractions as joining and separating parts that are referring to the same whole. Decompose fractions and mixed numbers into unit fractions and whole numbers, and express mixed numbers as a sum of unit fractions.
Taking Wholes Apart, Putting Parts Together	Visual Fraction Models	NF.2, NF.1, NF.3, NF.5, NF.6, NF.7: Use different ways of seeing and visualizing fractions to compare fractions using student generated visual fraction models. Use $>$, $<$ and $=$ to compare fraction size, through linear and area models, and determine whether fractions are greater or less than benchmark numbers, such as $\frac{1}{2}$ and 1.
Taking Wholes Apart, Putting Parts Together & Discovering Shape and Space	Circles, Fractions & Decimals	NF.5, NF.6, NF.7, OA.1. MD2, MD5, MD7: Understand, compare, and visualize fractions expressed as decimals. Recognize fractions with denominators of 10 and 100, e.g., 25 cents can be written as 0.25 or 25/100. Connect a circle fraction model to the clock face. Example $\frac{3}{10} + \frac{4}{100} = \frac{30}{100} + \frac{4}{100} = \frac{34}{100}$
Discovering Shape and Space	Shapes & Symmetries	MD.5, MD.6, MD.7, G.1, G.2, G.3, NBT.3, NBT.4, Draw and identify shapes, looking at the relationships between rays, lines, and angles. Explore symmetry through folding activities.
Discovering Shape and Space	Connected Problem Solving	OA.3, MD.1, MD.2, OA2, MD.3, NBT.3 place value, NBT.4, NBT.5, NBT.6, OA.2, OA.3, G.3: Solve problems with perimeter, area, volume, distance, and symmetry, using operations and measurement.

Grade Five

In fifth grade, equivalence and flexibility are big ideas, with both particularly relating to operations and fractions. Using relationships in the world to make meaning out of multiplication, division, fractions, and estimation requires a great deal of exploration. Using portion sizes to estimate with fractions is helpful because thinking about portions is a useful and underdeveloped idea that gives fractions meaning and utility. At this grade, students work with powers of ten, use exponential notation, and can explain patterns in the placement of the decimal point when a decimal is multiplied by a power of 10.

Fifth-grade students are expected to fully understand the place value system, including decimal values to thousandths, building from the foundation laid in earlier grades. Ideas to help with decimal understanding include using base ten blocks, with the 3-dimensional cube representing one unit so that students have a tactile, visual model to consider the value of the small cube,

the rod, and the 10 by 10 flat. In a virtual environment, students can use a CAD, Tinkercad, or other program to design and build complex shapes. While they are building 3-D representations in a 2-D space, it is important to ask students to think about what makes their 2-D drawings appear to be 3-D. Shapes in this environment may appear to be a parallelogram or a rhombus when they are representing a 3-D object—what the shape of the face really is in 3-D space is a square. Asking students to build their CAD designs out of cardboard or paper is a good way to have them explore the way shapes look when they change the angle of their view.

Another useful tool is a printed 10 x 10 grid. Students visualize the whole grid as representing the whole and can shade in various decimal values. Fifth-grade students use equivalent fractions to solve problems; so, it is important that they have a strong grasp of equality and can use benchmark fractions (e.g., $1/2$, $2/3$, $3/4$) to reason about, compare, and calculate with fractions. Experiences with placing whole numbers, fractions, and decimals on the same number line contribute to building fraction number sense. Students need time and opportunity to collaborate, critique, and reason about where to place the numbers on the number line.

When students in fifth grade conduct data investigations, they ask questions, collect data, analyze results, and communicate their findings. While the data visualizations included in the fifth-grade standards only include picture graphs, bar graphs, and line plots, students do not need to be restricted to these; data in the modern world is represented in many creative and non-standard ways, and it is important that students learn to read such data representations. Also, while standard data representations, such as bar graphs, show repeated measurements of a *single* varying quantity, science curricula in particular, and many questions of interest in general, require the consideration of relationships between *two or more different* changing quantities, such as erosion and time (NGSS 4-ESS2-1 Earth's Systems) or length or direction of shadows and time (NGSS 5-ESS1-2 Earth's Place in the Universe). Such reasoning involves multiple variables, which is an important aspect of modern encounters with data that students experience. Although the scatter plot, a crucial data representation tool for two varying quantities, is not expected to be fully understood until later grades (8.SP.1), it can be explored informally much earlier for students to be prepared for middle school content. For example, students can plot quantities changing over time (e.g., height of a plant, length of the day, high temperature for the day), with time on the horizontal axis and the changing quantity on the vertical. Once such a plot is created, it is an excellent context for a “notice and wonder” discussion.

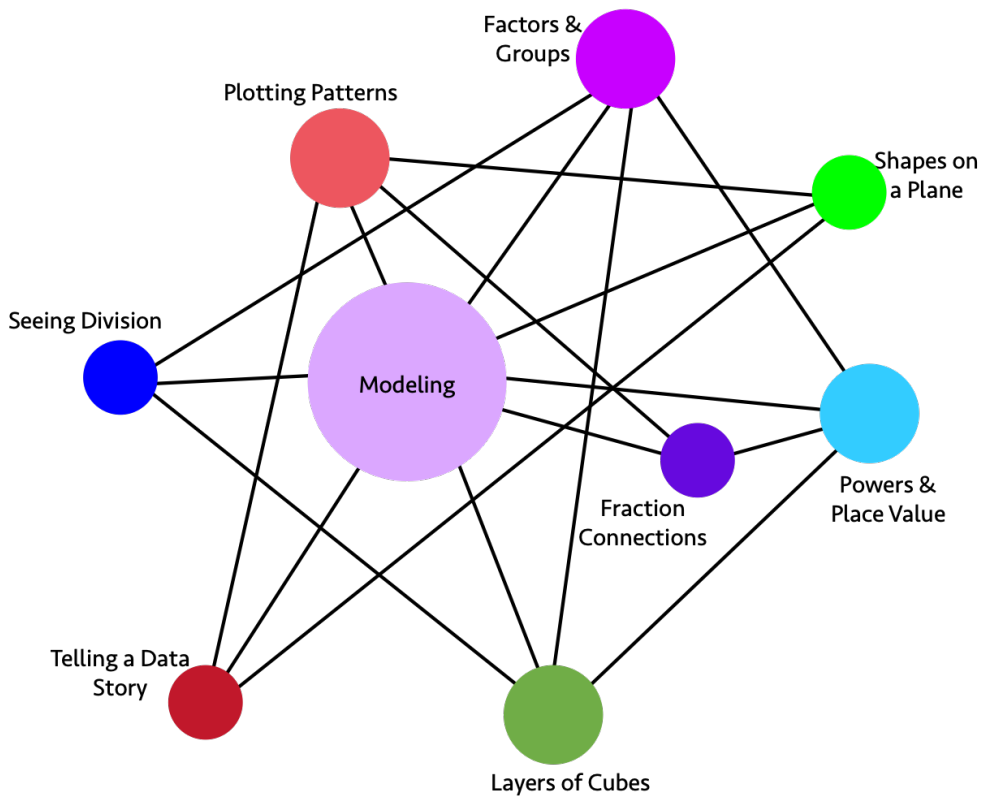
Moving to the fore in fifth grade are ideas about patterns and relationships in two- and three-dimensional space. Students begin to use the coordinate plane to represent and question relationships, and they begin to think about how to count and represent volume using cubic units. Providing investigations where students see volume as a visual model of unit cubes contained inside a 3-D shape is important work. Students have ample opportunities to study the volume of complex shapes, e.g., a pyramid, where they construct the volume as layers of unit cubes and grapple with the fractions of unit cubes that make up the volume. This is a way to further connect the meaning of a unit with fractions of a unit. In a virtual environment, students can build complex shapes in Tinkercad or other similar apps. Ask students to create and then hand draw their designs since this will engage different areas of their brain. If students

are building complex 3-D designs virtually in a CAD (Computer-Aided Design) space, it is a good idea to ask them to try to construct their designs outside of the computer environment. Ask if they can construct their shape out of cardboard or paper. Was it possible to build the shapes they had drawn? What challenges did you face?

Students who are English learners are encouraged to use their developing English and native language assets (e.g., cognates, morphological awareness) and draw on their prior knowledge. Teachers can examine text and tasks for key language forms and structures, and general as well as discipline-specific, high-utility academic vocabulary words linked to the big ideas and connections. Teachers can provide purposeful scaffolds and supports to engage EL students in sustained mathematical discourse in multiple contexts to build academic language and knowledge. Planned and “just-in-time” scaffolds and supports provide multiple entry points for meaning making and sharing of ideas in mathematical ways and include representations, expression starters and builders, and targeted and high-utility academic vocabulary and language structures (e.g., problems, explanations, arguments, descriptions, procedures). Teachers guide deconstruction and/or co-construction of problems, investigations, arguments, explanations, descriptions, methods, and connections.

Critical Areas of Instructional Focus

Figure 7.4. Grade 5 Big Ideas



Long description: The graphic illustrates the connections and relationships of some fifth-grade mathematics concepts. Direct connections include:

- Factors & Groups directly connects to: Powers & Place Values, Layers of Cubes, Modeling, Seeing Division
- Shapes on a Plane directly connects to: Telling a Data Story, Modeling, Plotting Patterns
- Powers & Place Value directly connects to: Layers of Cubes, Fraction Connections, Modeling, Factors & Groups
- Layers of Cubes directly connects to: Powers & Place Value, Factors & Groups, Modeling, Seeing Division
- Telling a Data Story directly connects to: Shapes on a Plane, Modeling, Plotting Patterns

- Seeing Division directly connects to: Layers of Cubes, Modeling, Factors & Groups
- Plotting Patterns directly connects to: Telling a Data Story, Modeling, Fraction Connections, Shapes on a Plane
- Fraction Connections directly connects to: Powers & Place Value, Modeling, Plotting Patterns
- Modeling directly connects to: Plotting Patterns, Factors & Groups, Shapes on a Plane, Powers & Place Value, Fraction Connections, Layers of Cubes, Telling a Data Story, Seeing Division

Figure 7.4a. Grade 5 Content Connections, Big Ideas, and Standards

Content Connection	Big Idea	Grade 5 Standards
Communicating Stories with Data	Plotting Patterns	G.1, G.2, OA.3: MD.2, NF.7: Students generate and analyze patterns, plotting them on a line plot or coordinate plane, and use their graph to tell a story about the data. Some situations should include fraction and decimal measurements, such as a plant growing.
Communicating Stories with Data & Exploring Changing Quantities & Discovering Shape & Space	Telling a Data Story	G.1, G.2, OA.3: Understand a situation, graph the data to show patterns and relationships, and to help communicate the meaning of a real-world event.
Exploring Changing Quantities	Factors & Groups	OA.1, OA.2, MD.4, MD.5: Students use grouping symbols to express changing quantities and understand that a factor can represent the number of groups of the quantity.
Exploring Changing Quantities	Modeling	NBT.3, NBT.5, NBT.7, NF.1, NF.2, NF.3, NF.4, NF.5, NF.6, NF.7, MD.4, MD.5, OA.3: Set up a model and use whole, fraction, and decimal numbers and operations to solve a problem. Use concrete models and drawings and justify results.

Content Connection	Big Idea	Grade 5 Standards
Exploring Changing Quantities & Taking Wholes Apart, Putting Parts Together	Fraction connections	NF.1, NF.2, NF.3, NF.4, NF.5, NF.7, MD.2, NBT.3: Make and understand visual models, to show the effect of operations on fractions. Construct line plots from real data that include fractions of units.
Taking Wholes Apart, Putting Parts Together	Seeing Division	MD.3, MD.4, MD.5, NBT.4, NBT.6, NBT.7: Solve real problems that involve volume, area, and division, setting up models and creating visual representations. Some problems should include decimal numbers. Use rounding and estimation to check accuracy and justify results.
Taking Wholes Apart, Putting Parts Together	Powers and Place Value	NBT.3, NBT.2, NBT.1, OA.1, OA.2: Use whole number exponents to represent powers of 10. Use expanded notation to write decimal numbers to the thousandths place and connect decimal notation to fractional representations, where the denominator can be expressed in powers of 10.
Discovering Shape and Space	Layers of Cubes	MD.5, MD.4, MD.3, OA.1, MD.1: Students recognize volume as an attribute of three-dimensional space. They understand that a 1-unit by 1-unit by 1-unit cube is the standard unit for measuring volume. They decompose three-dimensional shapes and find volumes of right rectangular prisms by viewing them as decomposed into layers of arrays of cubes.
Discovering Shape & Space & Exploring Changing Quantities	Shapes on a Plane	G.1, G.2, G.3, G4, OA.3, NF.4, NF.5, NF.6: Graph 2-D shapes on a coordinate plane, notice and wonder about the properties of shapes, parallel and perpendicular lines, right angles, and equal length sides. Use tables to organize the coordinates of the vertices of the figures and study the changing quantities of the coordinates.

Chapter 8: Mathematics in Grades Six through Eight

Figure 8.1. A Progression Chart of Big Ideas through Grades 6-8

Content Connections	Big Ideas: Grade 6	Big Ideas: Grade 7	Big Ideas: Grade 8
Communicating Stories with Data	Variability in data	Visualize Populations	Data explorations
Communicating Stories with Data	The shape of distributions	Populations and samples	Data graphs and tables
Communicating Stories with Data	n/a	Probability Models	Interpret scatter plots
Exploring Changing Quantities	Fraction relationships	Proportional Relationships	Multiple representations of functions
Exploring Changing Quantities	Patterns inside numbers	Unit rates in the world	Linear equations
Exploring Changing Quantities	Generalizing with multiple representations	Graphing relationships	Slopes and intercepts
Exploring Changing Quantities	Relationships between variables	Scale Drawings	Interpret scatter plots
Taking Wholes Apart, Putting Parts Together	Model the world	Shapes in the world	Cylindrical investigations
Taking Wholes Apart, Putting Parts Together	Nets and Surface Area	2-D and 3-D connections	Pythagorean explorations
Taking Wholes Apart, Putting Parts Together	n/a	Angle relationships	Big and small numbers
Discovering shape and space	Nets and Surface Area	Shapes in the world	Shape, number, and expressions
Discovering shape and space	Distance and direction	2-D and 3-D connections	Pythagorean explorations
Discovering shape and space	Graphing shapes	Scale drawings	Cylindrical investigations
Discovering shape and space	n/a	Angle relationships	Transformational geometry

As students enter the middle grades, the number sense they acquired in the elementary grades deepens with the content. Students transition from exploring numbers and arithmetic operations in K–5 to exploring relationships between numbers (CC2—Exploring Changing Quantities and CC3—Taking Wholes Apart and Putting Parts Together) and making sense of contextual situations using various representations. SMP.2 is especially critical at this stage, as students represent a wide variety of real-world situations through the use of real numbers and variables in expressions, equations, and inequalities. Big ideas in number sense for the middle grades include:

- number line understanding;
- proportions, ratios, percents, and relationships among these; and
- and generalized numbers as leading to algebra.

The big ideas of data science include the following:

- Data in the world: exploration, interpretation, decision making, ethics
- Variability: describing, displaying, and comparing
- Sampling to understand a population: randomness, bias, how many?
- Are they related? Multivariate thinking
- What are the chances? Probability as the basis for data-based claims

As in earlier grades, students experience data science as a tool to help understand their worlds via a process that begins with wondering questions. This is also the beginning of the mathematical modeling cycle, the statistical and data science exploration process, and investigations in science.

The sixth through eighth grade span is an important time for further development of important mathematical concepts needed for high school. Students are introduced to irrational numbers through investigations using the Pythagorean Theorem. Students work with right triangles and apply their learning to further investigations of plane figures and solids, where the Pythagorean Theorem is useful in finding unknown measures. Students explore cylinders, cones, and spheres, while noticing radius as a useful component of right triangles. Students continue investigating 3-D shapes as they consider these shapes to be made up of slices of 2-D shapes. Students begin their formal study of transformational geometry as the study of shapes that twist, turn, and grow in the plane. Students investigate and make meaning of these transformations as they connect them to similarity and congruence.

The following interview highlights an educator who is using digital tools to help students in this grade span express their thinking and provide feedback and supports as necessary.

Voices from the Field: Martin Joyce | Taylor Middle School | Millbrae, CA

Joyce—a 12-year veteran who currently teaches pre-algebra at Taylor Middle School—leverages technology integration, collaboration, and feedback to engage all of his learners.

Describe some of the challenges that you and your students have experienced with the implementation of distance learning. How have you turned those into opportunities for success?

In terms of success, the most prominent example for me has been around using Desmos—imagine having an interactive PowerPoint with graphs, sketch capabilities, and fantastic feedback options. I've been fortunate to be part of their pilot sixth through eighth grade math curriculum. Students get real-time feedback and can use the data to continually revise their work. We can take snapshots of students' work and use those as models to demonstrate to the entire class. In other words, it's not just me showing them the successful way to problem solve.

Desmos complements monitoring, selecting, sequencing, and connecting extremely well. I really focus on questions to support those students who may struggle. As I monitor their work in real time, I see what they are doing (sketching) and then identify whose work I want to share. I select and invite participation from diverse learners who may not just volunteer. When it comes to their problem solving, I often start with the most common mistakes or the methods of success.

In terms of challenges, the primary one for me has been around pacing. I think teachers have had to accept that we are often not going to get as much done as we had in years past. I've had to continually ask myself reflective questions such as, "What do I skip?" or "Where do I compress?" I've really had to allow for more time, and I've had to learn to be more patient. I've also had to be flexible with when and how students may respond to questions.

Another challenge is that some students don't participate in places like virtual breakout rooms versus how they might in face-to-face environments. Overall, I think the student discourse has lessened during distance learning. I've had to intervene and facilitate more. I've had to prompt them more to connect with one another. I've had to teach them more communication and collaboration skills.

One small tip I recently picked up is facilitating a Zoom [video conferencing platform] chat waterfall. I ask students to answer a question in the chat box but instruct them to not hit enter until told to do so. I honor time to wait or think and then have them hit enter to create a cascade (a waterfall) of responses all at once. Otherwise, we are influenced by one another's responses or thoughts.

How do you strike a balance between analog and digital tools or synchronous and asynchronous learning experiences for your students?

This year, our school has scheduled asynchronous time Tuesday through Friday, 1:30-3:00 pm. Our classes are 80 minutes long. Students have three of these classes per day. We created the asynchronous opportunities to address the amount of screen time and to develop independence. We have used this time for office hours focused on intervention and support. For math, I take the practice problems and do a screencast recording for these asynchronous times. Although there are plenty of great YouTube tutorial videos for math, I think it's valuable for students to have access to videos I create.

Synchronous time for me is focused on instruction and students working with me in real time. I use warm-ups to activate prior knowledge, we do some problems together with discussion, and then we have lesson synthesis after all of the activities. I have started to incorporate cool-downs or exit tickets. I use one or two problems for quick formative assessments.

How are you using technology to help students build foundational algebra skills and understand key concepts?

Desmos and game-like applications are helpful. Technology allows students to see these math principles in concrete representations. It allows them to visualize. They can try things and see if they are right or wrong. This works well with both horizontal and vertical number lines.

In eighth grade, we work with ratios and the slope of a graph. Desmos would be the key here for the graphing. Google Sheets [collaborative online spreadsheets] work well for the percentages and two-way tables.

I have a year-end project on the Pythagorean Theorem, which has traditionally been challenging for students who have been absent. I now use Edpuzzle [video-based lessons] to record myself and share videos on how they can get started. It's great to voice over (audio record) my instruction and demonstrations.

What are your main priorities or concerns when selecting technology for your classes?

My primary considerations are both access and ease of use. I'm really starting to think that less is more when it comes to educational technology. I like to have a baseline app or an interface that I can use. It can't always be about adding an entirely new thing.

For example, Flipgrid [video-based discussion software] is a great add-on, and it works extremely well with Desmos. It's easy to use, and it creates opportunities for student voice. I use it from the first days of school, where students record themselves demonstrating the correct pronunciation of their name, as well as throughout the year for demonstrations of mastery. I will still use Desmos for the assessment, but if a student wants to increase their score, they can record themselves in Flipgrid discussing their mistakes and how to arrive at the right answer. I want them to convince me what they know now and didn't know before. It doesn't matter when one knows it, just that one knows it. I want to value the work and the learning.

I've really found HyperDocs [digital lesson plans] to be especially helpful during distance learning and think it will continue to have significant value as we return to face-to-face. It really gives students a structure to follow when doing a multi-step project.

There are a lot of great tools out there, but I don't want to overwhelm my students with too many. I think teachers need to be more careful and intentional with introducing new technologies. I try to have a core application with a couple great add-ons.

Grade Six

Proportional reasoning, unit rates, and generalizing relationships are central to sixth grade. This represents a major shift for students and is worthy of deep, sustained attention. Students build new ways to represent the world symbolically, on the number line, and through data that add nuance to the mathematical terrain. In sixth grade, students are introduced to the concepts of ratios and unit rates, and they use tables of equivalent ratios, double number lines, tape diagrams, and equations to solve real-world problems. A critical feature to emphasize for students is the ability to think multiplicatively, as well as additively.

Students are **often** introduced to the idea of a variable, not through the concept of variation, but through exercises that ask them to find a missing number that is represented by x or another variable. Unfortunately, this gives them the idea that a variable stands for a single number, rather than something that varies - which causes students problems when they later need to learn about functions and other uses of algebra where a variable varies. The best way to introduce students to the idea of a variable is to give them examples of pattern growth that they can analyze, represent in words, and eventually as variables. The Path problem—finding how many squares are in the path that borders different sized squares—is an ideal way to introduce the concept of a variable.¹² Ideas of equivalence and operations, laid before in earlier grades, now take on new meaning, as students apply properties of operations to generate equivalent expressions and identify when two expressions are equivalent.

Sixth-grade students engage in data investigations to help them understand variability. If they are given opportunities to develop curiosity and ask questions about the world, they can collect and analyze data, determine and confirm results, and represent findings with different representations. Teachers can ask students to collect data and/or bring in real data sets from the world that students are invited to investigate. The Common Online Data Analysis Platform (CODAP) is a website providing free educational software for data analysis. CODAP is an ideal tool for introducing students to data exploration. CODAP includes many interesting data sets and lessons, and students can look visually at the shape of data distributions, leading to consideration of measures of center and variability.¹³

¹² <https://www.youcubed.org/algebra/-week-1>

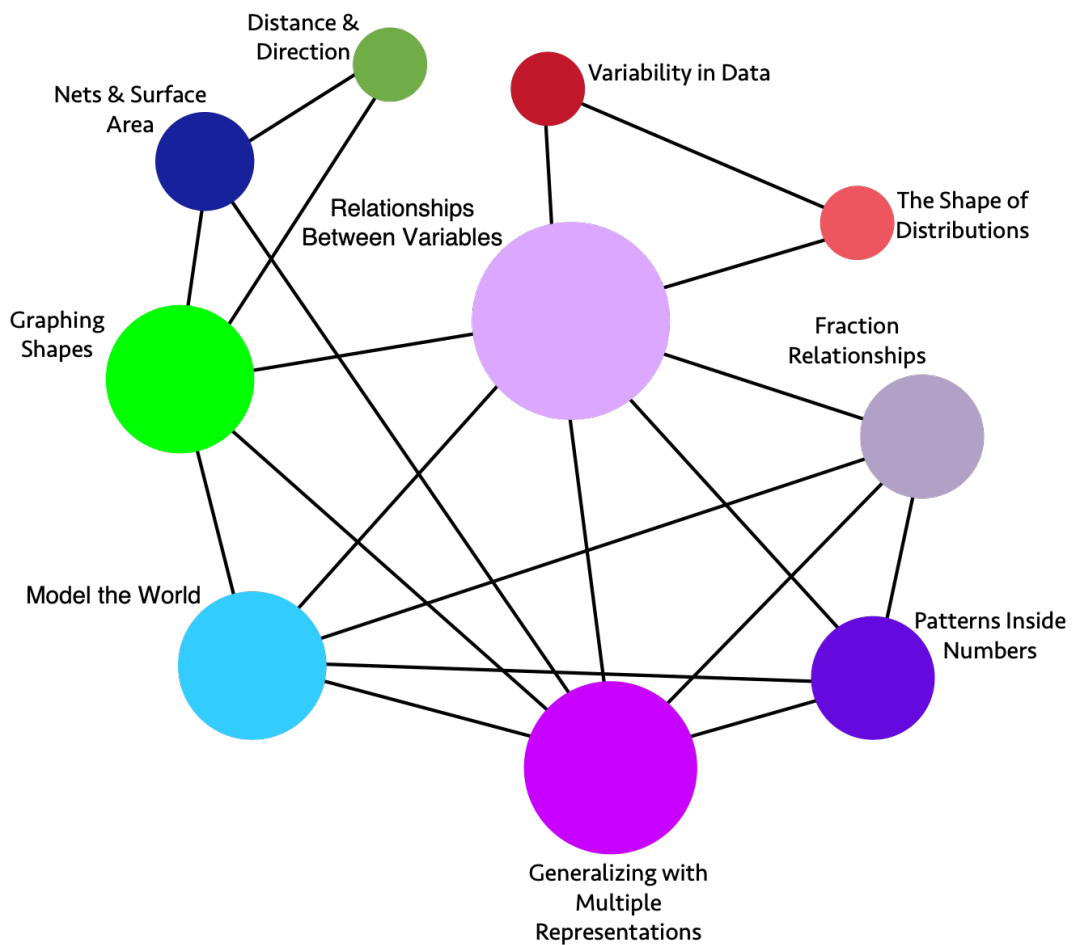
¹³ <https://codap.concord.org/>

Students in grade six also develop new ways to compose and decompose with two- and three-dimensional shapes, thinking about volume and area as additive and using nets to explore the surfaces that create solids. Moving from 3-D solids to 2-D representations of 3-D solids is a topic where students explore, construct, and take apart, building with unit cubes and drawing representations. In a virtual environment, students can use a CAD, Tinkercad, or other programs to design and build complex shapes. While they are building 3-D representations in a 2-D space, it is important to ask students to think about what makes their 2-D drawings appear to be 3-D. Shapes in this environment may appear to be a parallelogram or a rhombus when they are representing a 3-D object—what the shape of the face really is in 3-D space is a square. Asking students to build their CAD designs out of cardboard or paper is a good way to have them explore the way shapes look when they change the angle of their view. Students in grade six also learn about absolute value. An ideal opportunity to learn absolute value is an exploration of the shapes on a 4-quadrant coordinate grid, with absolute value used as a measure of distance, while integer coordinates represent the vertices of the shapes.

Students who are English learners are encouraged to use their developing English and native language assets (e.g., cognates, morphological awareness) and draw on their prior knowledge. Teachers can examine text and tasks for key language forms and structures, and general as well as discipline-specific, high-utility academic vocabulary words linked to the big ideas and connections. Teachers can provide purposeful scaffolds and supports to engage EL students in sustained mathematical discourse in multiple contexts to build academic language and knowledge. Planned and “just-in-time” scaffolds and supports provide multiple entry points for meaning making and sharing of ideas in mathematical ways and include representations, expression starters and builders, and targeted and high-utility academic vocabulary and language structures (e.g., problems, explanations, arguments, descriptions). Teachers guide deconstruction and/or co-construction of problems, investigations, arguments, explanations, descriptions, and connections.

Critical Areas of Instructional Focus

Figure 8.2. Grade 6 Big Ideas



Long description: The graphic illustrates the connections and relationships of some sixth-grade mathematics concepts. Direct connections include:

- Variability in Data directly connects to: The Shape of Distributions, Relationships Between Variables
- The Shape of Distributions directly connects to: Relationships Between Variables, Variability in Data
- Fraction Relationships directly connects to: Patterns Inside Numbers, Generalizing with Multiple Representations, Model the World, Relationships Between Variables
- Patterns Inside Numbers directly connects to: Fraction Relationships, Generalizing with Multiple Representations, Model the World, Relationships Between Variables

- Generalizing with Multiple Representations directly connects to: Patterns Inside Numbers, Fraction Relationships, Model the World, Relationships Between Variables, Nets & Surface Area, Graphing Shapes
- Model the World directly connects to: Fraction Relationships, Relationships Between Variables, Patterns Inside Numbers, Generalizing with Multiple Representations, Graphing Shapes
- Graphing Shapes directly connects to: Model the World, Generalizing with Multiple Representations, Relationships Between Variables, Distance & Direction, Nets & Surface
- Nets & Surface directly connects to: Graphing Shapes, Generalizing with Multiple Representations, Distance & Direction
- Distance & Direction directly connects to: Graphing Shapes, Nets & Surface Area
- Relationships Between Variables directly connects to: Variability in Data, The Shape of Distributions, Fraction Relationships, Patterns Inside Numbers, Generalizing with Multiple Representations, Model the World, Graphing Shapes

Figure 8.2a. Grade 6 Content Connections, Big Ideas, and Standards

Content Connection	Big Idea	Grade 6 Standards
Communicating Stories with Data	Variability in Data	SP.1, SP.5, SP.4: Investigate real world data sources, ask questions of data, start to understand variability - within data sets and across different forms of data, consider different types of data, and represent data with different representations.
Communicating Stories with Data	The Shape of Distributions	SP.2, SP.3, SP.5: Consider the distribution of data sets - look at their shape and consider measures of center and variability to describe the data and the situation which is being investigated.
Exploring Changing Quantities	Fraction Relationships	NS.1, RP.1, RP.3: Understand fractions divided by fractions, thinking about them in different ways (e.g., how many $\frac{1}{3}$ are inside $\frac{2}{3}$?), considering the relationship between the numerator and denominator, using different strategies and visuals. Relate fractions to ratios and percentages.
Exploring Changing Quantities	Patterns inside Numbers	NS.4, RP.3: Consider how numbers are made up, exploring factors and multiples, visually and numerically.

Content Connection	Big Idea	Grade 6 Standards
Exploring Changing Quantities	Generalizing with Multiple Representations	EE.6, EE.2, EE.7, EE.3, EE.4, RP.1, RP.2, RP.3: Generalize from growth or decay patterns, leading to an understanding of variables. Understand that a variable can represent a changing quantity or an unknown number. Analyze a mathematical situation that can be seen and solved in different ways and that leads to multiple representations and equivalent expressions. Where appropriate in solving problems, use unit rates.
Exploring Changing Quantities	Relationships Between Variables	EE.9, EE.5, RP.1, RP.2, RP.3, NS.8, SP.1, SP.2: Use independent and dependent variables to represent how a situation changes over time, recognizing unit rates when it is a linear relationship. Illustrate the relationship using tables, 4 quadrant graphs and equations, and understand the relationships between the different representations and what each one communicates.
Taking Wholes Apart, Putting Parts Together	Model the World	NS.3, NS.2, NS.8, RP.1, RP.2, RP.3: Solve and model real world problems. Add, subtract, multiply, and divide multi-digit numbers and decimals, in real-world and mathematical problems - with sense making and understanding, using visual models and algorithms.
Taking Wholes Apart, Putting Parts Together & Discovering Shape and Space	Nets and Surface Area	EE.1, EE.2, G.4, G.1, G.2, G.3: Build and decompose 3-D figures using nets to find surface area. Represent volume and area as expressions involving whole number exponents.
Discovering Shape and Space	Distance and Direction	NS.5, NS.6, NS.7, G.1, G.2, G.3, G.4: Students experience absolute value on numbers lines and relate it to distance, describing relationships, such as order between numbers using inequality statements.
Discovering Shape and Space	Graphing Shapes	G.3, G.1, G.4, NS.8, EE.2: Use coordinates to represent the vertices of polygons, graph the shapes on the coordinate plane, and determine side lengths, perimeter, and area.

Grade Seven

A big idea for seventh grade is proportional reasoning, which students experience in many different ways as they consider fractions, decimals, percents, and integers. An important idea for students is that every fraction, decimal, percent, integer, and whole number can be written

as a rational number defined to be the ratio of two integers and understandings of fractions, decimals, percents, integers, and whole numbers can all be subsumed into a larger understanding of rational numbers. This unified understanding is achieved, in part, through students' use of number lines to represent operations on rational numbers, such as the addition and subtraction of rational numbers on a number line. Students can be introduced to a host of representations as they reason through proportional situations: graphs, equations, verbal descriptions, tables, charts, and double number lines. There are many approaches to solving proportions, and it is important to emphasize that sense-making is more important than answer finding.

Students in seventh grade should continue investigations that involve generalization, allowing them to see and use algebra as a useful problem-solving tool.

A big idea in the Communicating Stories with Data strand is variability, and understanding variability is at the heart of data literacy. When working with visualizations of data, students consider not only the most popular value in a dataset (the mode) but also describe the shape and spread of data distributions. As they engage in experiences where they produce their own data through measurement, teachers can highlight for students the variation that results. For example, if students plant a particular variety of flower seed at multiple locations around the school, then measure the plants' height and the amount of sunlight each month, they can conduct investigations into the ways plant growth and sunlight relate to each other. They discuss and describe any patterns in their bivariate data and reasons for the variability. Finally, students consider their own measurement techniques and how confident they are that they all measured the same way (so that if someone else measured, they would get the same height or sunlight). Students can be invited to study populations by taking random samples and determining if the samples accurately represent the population, considering issues of bias and ethics. They can use classroom simulations and computer software to model repeated sampling, analyzing the variation in results. Students can also use measures of center and variability to draw comparative inferences about populations, considering what the visual plots show.

New ideas for grade seven are randomness, probability, and uncertainty. At this point, students can begin to conceive of probability as a measure of the chance that something will happen, seeing it as a basic measure of certainty or uncertainty. They can learn to use sample spaces, lists, tables, and tree diagrams.

Students connect proportional reasoning to the two- and three-dimensional world through the construction of scale figures. Students investigate angles and connections between angles, including supplementary and complementary angles, noticing that increasing one decreases the other.

Students solve problems involving solid figures and develop intuition about 2-D slices of 3-D figures. The idea of sliced objects and the shape produced, as well as measure and area, is important to the further study of mathematics in higher education. Students can be using technology to support their calculations as they are building understanding of the physical

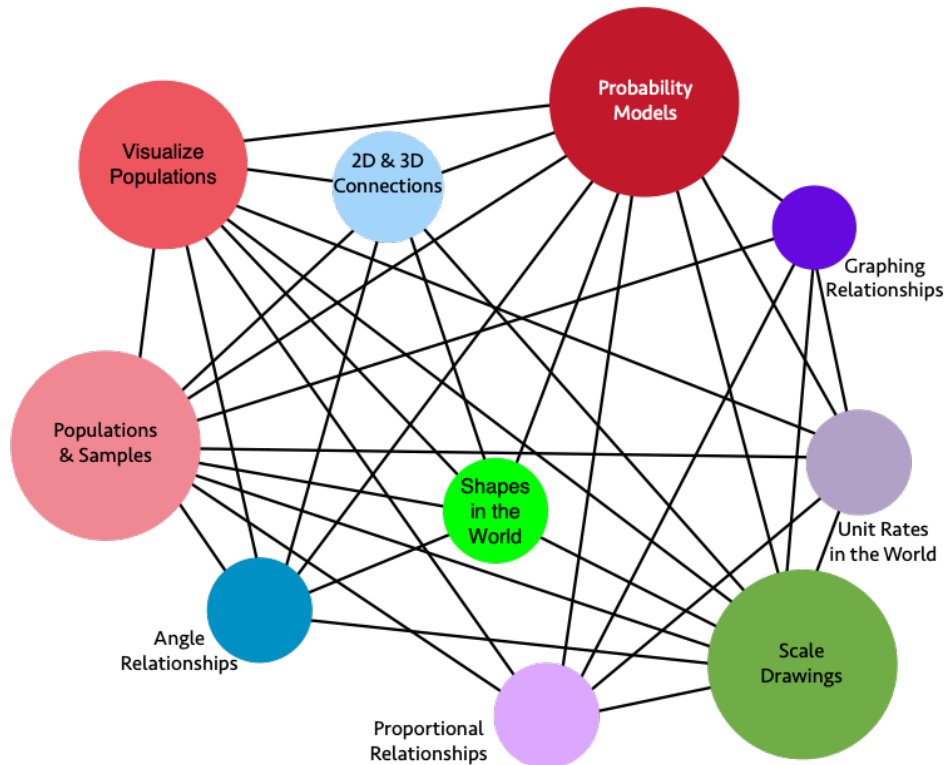
shapes. In a virtual environment, students can use a CAD, Tinkercad, or other program to design and build complex shapes. While they are building 3-D representations in a 2-D space, it is important to ask students to think about what makes their 2-D drawings appear to be 3-D. Shapes in this environment may appear to be a parallelogram or a rhombus when they are representing a 3-D object—what the shape of the face really is in 3-D space is a square. Asking students to build their CAD designs out of cardboard or paper is a good way to have them explore the way shapes look when they change the angle of their view.

Measurements that include decimal and fraction numbers can be used throughout their investigations. When considering physical objects and quantifying the objects using measures, students are supported in building important connections between visual models and numbers.

Students who are English learners are encouraged to use their developing English and native language assets (e.g., cognates, morphological awareness) and draw on their prior knowledge. Teachers can examine text and tasks for key language forms and structures, and general as well as discipline-specific, high-utility academic vocabulary words linked to the big ideas and connections. Teachers can provide purposeful scaffolds and supports to engage EL students in sustained mathematical discourse in multiple contexts to build academic language and knowledge. Planned and “just-in-time” scaffolds and supports provide multiple entry points for meaning making and sharing of ideas in mathematical ways and include representations, expression starters and builders, and targeted and high-utility academic vocabulary and language structures (e.g., problems, explanations, arguments, descriptions, and connections). Teachers guide deconstruction and/or co-construction of problems, investigations, arguments, explanations, descriptions, and connections. Clear and precise expressions, as well as cohesive writing, support stronger communication of mathematical concepts and practices.

Critical Areas of Instructional Focus

Figure 8.3 Grade 7 Big Ideas



Long description: The graphic illustrates the connections and relationships of some seventh-grade mathematics concepts. Direct connections include:

- Angle Relationships directly connects to: Scale Drawings, 2D & 3D Connections, Populations & Samples, Proportional Relationships, Shapes in the World, Visualize Populations, Probability Models
- Scale Drawings directly connects to: 2D & 3D Connections, Graphing Relationships, Populations & Samples, Unit Rates in the World, Proportional Relationships, Visualize Populations, Probability Models, Angle Relationships
- Graphing Relationships directly connects to: Populations & Samples, Unit Rates in the World, Proportional Relationships, Probability Models, Scale Drawings
- 2D & 3D Connections directly connects to: Scale Drawings, Angle Relationships, Probability Models, Proportional Relationships, Visualize Populations, Shapes in the World, Populations & Samples

- Populations & Samples directly connects to: 2D & 3D Connections, Scale Drawings, Angle Relationships, Probability Models, Proportional Relationships, Visualize Populations, Shapes in the World, Unit Rates in the World, Graphing Relationships
- Unit Rates in the World directly connects to: Populations & Samples, Graphing Relationships, Scale Drawings, Proportional Relationships, Probability Models, Visualize Populations
- Shapes in the World directly connects to: Populations & Samples, 2D & 3D Connections, Proportional Relationships, Scale Drawings, Angle Relationships, Probability Models, Visualize Populations
- Visualize Populations directly connects to: 2D & 3D Connections, Scale Drawings, Angle Relationships, Probability Models, Proportional Relationships, Populations & Samples, Shapes in the World, Unit Rates in the World
- Probability Models directly connects to: 2D & 3D Connections, Scale Drawings, Angle Relationships, Proportional Relationships, Visualize Populations, Shapes in the World, Unit Rates in the World, Graphing Relationships, Populations & Samples
- Proportional Relationships directly connects to: 2D & 3D Connections, Scale Drawings, Angle Relationships, Probability Models, Populations & Samples, Visualize Populations, Shapes in the World, Unit Rates in the World, Graphing Relationships

Figure 8.3a. Grade 7 Content Connections, Big Ideas, and Standards

Content Connection	Big Idea	Grade 7 Standards
Communicating Stories with Data	Populations & Samples	<p>SP.1, SP.2, RP.1, RP.2, RP.3, NS.1, NS.2, NS.3, EE.3: Study a population by taking random samples and determine if the samples accurately represent the population.</p> <ul style="list-style-type: none"> • Analyze and critique reports by examining the sample and the claims made to the general population • Use classroom simulations and computer software to model repeated sampling, analyzing the variation in results.
Communicating Stories with Data	Visualize Populations	<p>SP.3, SP.4, NS.1, NS.2, NS.3, EE.3: Draw comparative inferences about populations - consider what visual plots show, and use measures of center and variability</p> <ul style="list-style-type: none"> • Students toggle between the mathematical results and their meaningful interpretation with their given context, considering audiences, implications, etc.

Content Connection	Big Idea	Grade 7 Standards
Communicating Stories with Data	Probability Models	<p>SP.5, SP.6, SP.7, SP.8, RP.1, RP.2, RP.3, NS.1, NS.2, NS.3, EE.3: Develop a probability model and use it to find probabilities of events and compound events, representing sample spaces and using lists, tables, and tree diagrams.</p> <ul style="list-style-type: none"> • Compare observed probability and expected probability. • Explore potential bias and over-representation in real world data sets, and connect to dominating narratives and counter narratives used in public discourse.
Exploring Changing Quantities	Proportional Relationships	<p>EE.2, EE.3, RP.1, RP.2, RP.3: Explore, understand, and use proportional relationships: - using fractions, graphs, and tables.</p>
Exploring Changing Quantities	Unit Rates in the World	<p>RP.1, RP.2, RP.3, EE.1, EE.2, EE.3, EE.4: Solve real world problems using equations and inequalities, and recognize the unit rate within representations.</p>
Exploring Changing Quantities	Graphing Relationships	<p>EE.4, RP.1, RP.2, RP.3: Solve problems involving proportional relationships that can lead to graphing using geometry software and making sense of solutions.</p>
Taking Wholes Apart, Putting Parts Together & Discovering Shape and Space	2-D and 3-D Connections	<p>G.1, G.2, G.3, NS.1, NS.2, NS.3: Draw and construct shapes, slice 3-D figures to see the 2-D shapes. Compare and classify the figures and shapes using area, surface area, volume, and geometric classifications for triangles, polygons, and angles. Make sure to measure with fractions and decimals, using technology for calculations.</p>
Taking Wholes Apart, Putting Parts Together & Discovering Shape and Space	Angle Relationships	<p>G.5, G.6, NS.1, NS.2, NS.3: Explore relationships between different angles, including complementary, supplementary, vertical, and adjacent, recognizing the relationships as the measures change. For example, angles A and B are complementary. As the measure of angle, A increases, the measure of angle B decreases.</p>

Content Connection	Big Idea	Grade 7 Standards
Discovering Shape and Space & Exploring Changing Quantities	Scale Drawings	G.1, EE.2, EE.3, EE.4, NS.2, NS.3, RP.1, RP.2, RP.3: Solve problems involving scale drawings and construct geometric figures using unit rates to accurately represent real world figures. (Use technology for drawing)
Discovering Shape and Space & Exploring Changing Quantities	Shapes in the World	G.1, G.2, G.3, G.4, G.5, G.6, NS.1, NS.2, NS.3: Solve real life problems involving triangles, quadrilaterals, polygons, cubes, right prisms, and circles using angle measures, area, surface area, and volume.

Grade Eight

In eighth grade, students' understanding of rational numbers is extended in two important ways. First, rational numbers have decimal expansions, which eventually repeat, and vice versa. All numbers with decimal expansions, which eventually repeat, are rational numbers. A typical task to demonstrate the first aspect of this standard is to ask students to investigate long division with a calculator, or other technology, to demonstrate that $\frac{3}{11}$ has a repeating decimal expansion, and to explain why. As students realize the connection between the remainder and the repeating portion, their understanding of rational numbers can fully integrate with their understanding of decimals and place value.

Second, as students begin to recognize that there are numbers that are not rational, which are called *irrational* numbers, they can see that these new types of numbers can still be located on the number line and can also be approximated by rational numbers. The foundation for this recognition is actually built through seventh-grade geometry explorations of the relationship between the circumference and diameter of a circle, and formalized into the formula for circumference, where the division of the circumference by the diameter for a given circle always results in a number a little larger than three, irrespective of the size of the circle. In exploring this quotient of circumference by diameter, students get a look at a decimal approximation for their first irrational number, pi. In eighth grade, the notation for numbers expands greatly, with the introduction of integer exponents and radicals to represent solutions of equations. Number sense plays a critical role in eighth grade, as students can check the accuracy of their answers with estimation. They can also use technological tools to work with place value and to express large and small numbers in scientific notation.

Proportional relationships continue to be a hub of mathematical thought in eighth grade, serving as a tool for thinking about patterns of growth, functions, and geometric

transformations. Functions are an important addition to the algebraic space in eighth grade. One big idea that challenges students' notions of clean, linear relationships, after all their work on functions and proportions, is the idea of extracting meaning from data. Data in the real world is rarely neat and lock-step; this is an important moment to develop a lens for looking at scatter plots and genuinely asking what relationships can be found.

Eighth grade students conduct data investigations that allow them to interpret bivariate and multivariate data. They also continue to visualize and represent *single-variable* data with dot plots, histograms, and box plots; use measures of center and spread to describe such distributions; and compare distributions from different populations or samples using these representations and statistics. Students also construct scatter plots, which show an association between two variables that is visually identifiable. Fitting a function to the data is the creation of a mathematical model. This work begins in eighth grade with visual fitting of a linear model. While the type of function that is used most frequently is a line (a linear function), students also need experiences with plotting associations that are clearly non-linear, as well as experimenting with fitting other types of functions (quadratic, exponential).

Any standard data software (including spreadsheets, Desmos, Geogebra, CODAP) will fit lines, quadratic functions, and exponential functions to given data. Students have experiences fitting lines and some other functions visually (by adjusting parameters on appropriate function types in graphing software) and using appropriate software tools, which perform the regression behind the scenes.

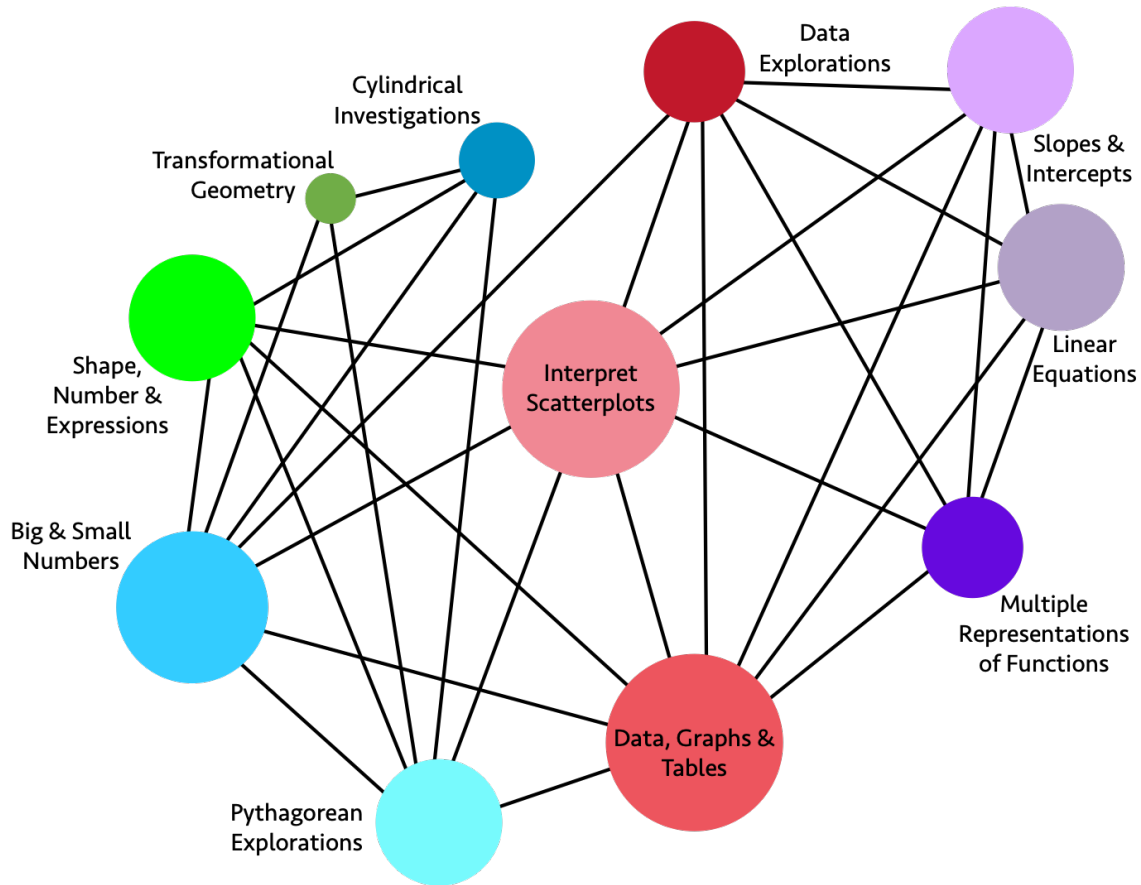
In grade eight, students are introduced to irrational numbers through the study of circles, spheres, and other solids that have a circle as a base. Students investigate the relationships between the side lengths of right triangles and use the Pythagorean Theorem to find a missing side length when two others are known. When studying quadrilaterals and using the Pythagorean Theorem, students can consider rectangles and squares with whole number side lengths and investigate which rectangles have diagonals with irrational side lengths. Students connect their understanding of right triangles and the Pythagorean Theorem to solids where they can use this knowledge to determine distances between two points. Students continue their study of two-dimensional shapes as they learn to move them across a plane, using transformations to investigate similarity and congruence. This initial journey into slides, rotations, reflections, and dilations is intended to be an initial introduction and should include the use of a dynamic geometric software.

Students who are English learners are encouraged to use their developing English and native language assets (e.g., cognates, morphological awareness) and draw on their prior knowledge. Teachers can examine text and tasks for key language forms and structures, and general as well as discipline-specific, high-utility academic vocabulary words linked to the big ideas and connections. Teachers can provide purposeful scaffolds and supports to engage EL students in sustained mathematical discourse in multiple contexts to build academic language and knowledge. Planned and "just-in-time" scaffolds and supports provide multiple entry points for meaning making and sharing of ideas in mathematical ways and include representations, expression starters and builders, and targeted and high-utility academic vocabulary and

language structures (e.g., problems, explanations, arguments, descriptions, and connections). Teachers guide deconstruction and/or co-construction of problems, investigations, arguments, explanations, descriptions, and procedures. Clear and precise expressions, as well as cohesive writing, support stronger communication of mathematical concepts and practices.

Critical Areas of Instructional Focus

Figure 8.4. Grade 8 Big Ideas



Long description: The graphic illustrates the connections and relationships of some eighth-grade mathematics concepts. Direct connections include:

- Data Explorations directly connects to: Slopes & Intercepts, Linear Equations, Multiple Representations of Functions, Data Graphs & Tables, Interpret Scatter plots, Big & Small Numbers

- Slopes & Intercepts directly connects to: Linear Equations, Multiple Representations of Functions, Data Graphs & Tables, Interpret Scatter plots, Data Explorations
- Linear Equations directly connects to: Slopes & Intercepts, Data Explorations, Multiple Representations of Functions, Data Graphs & Tables, Interpret Scatter plots
- Multiple Representations of Functions directly connects to: Data Graphs & Tables, Interpret Scatter plots, Data Explorations, Slopes & Intercepts, Linear Equations
- Data Graphs & Tables directly connects to: Multiple Representations of Functions, Linear Equations, Slopes & Intercepts, Data Explorations, Interpret Scatter plots, Shape Number & Expressions, Big & Small Numbers, Pythagorean Explorations
- Pythagorean Explorations directly connects to: Data Graphs & Tables, Interpret Scatter plots, Cylindrical Investigations, Transformational Geometry, Shape Number & Expressions, Big & Small Numbers
- Big & Small Numbers directly connects to: Pythagorean Explorations, Data Graphs & Tables, Interpret Scatter plots, Data Explorations, Cylindrical Investigations, Transformational Geometry, Shape Number & Expressions
- Shape Number & Expressions directly connects to: Big & Small Numbers, Pythagorean Explorations, Data Graphs & Tables, Interpret Scatter plots, Cylindrical Investigations
- Transformational Geometry directly connects to: Big & Small Numbers, Pythagorean Explorations, Cylindrical Investigations
- Cylindrical Investigations directly connects to: Big & Small Numbers, Pythagorean Explorations, Shape Number & Expressions, Transformational Geometry
- Interpret Scatter plots directly connects to: Data Explorations, Slopes & Intercepts, Linear Equations, Multiple Representations of Functions, Data Graphs & Tables, Pythagorean Explorations, Big & Small Numbers, Shape Number & Expressions

Figure 8.4a. Grade 8 Content Connections, Big Ideas, and Standards

Content Connection	Big Idea	Grade 8 Standards
Communicating Stories with Data & Exploring Changing Quantities	Interpret Scatter plots	SP.1, SP.2, SP.3, EE.2, EE.5, F.1, F.2, F.3: Construct and interpret data visualizations, including scatter plots for bivariate measurement data using two-way tables. Describe patterns noting whether the data appear in clusters, are linear or nonlinear, whether there are outliers, and if the association is negative or positive. Interpret the trend(s) in change of the data points over time.

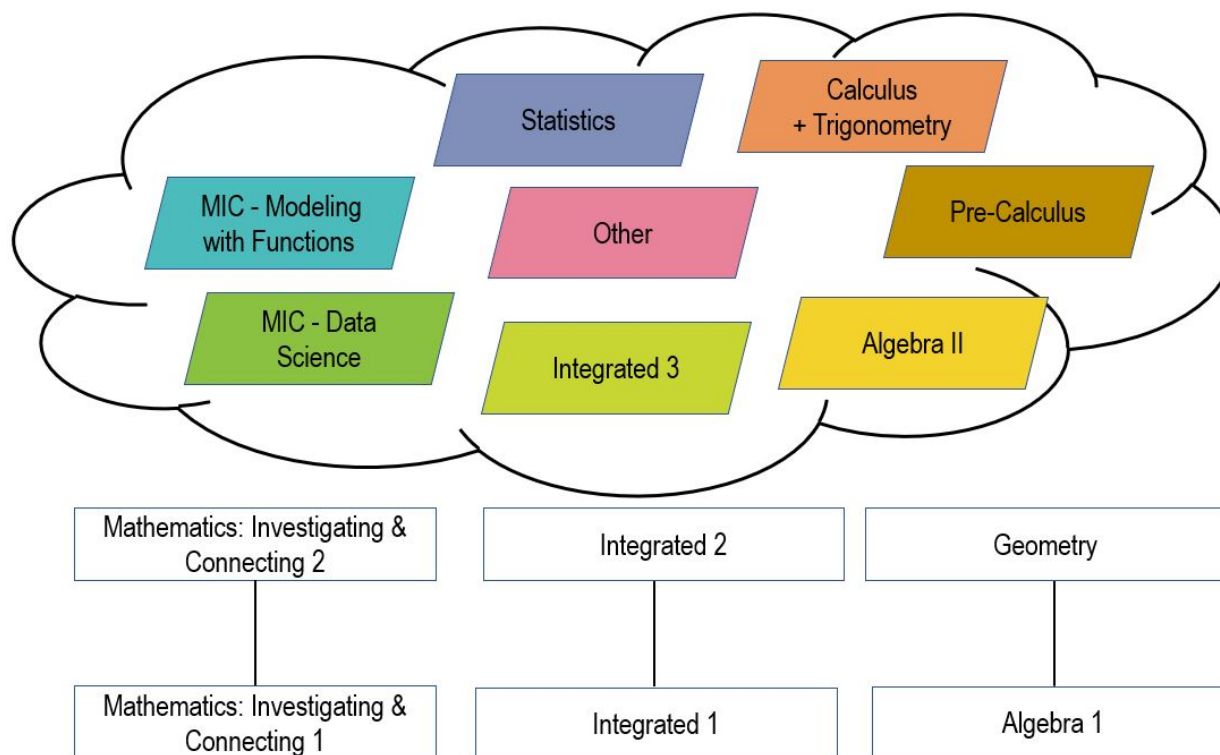
Content Connection	Big Idea	Grade 8 Standards
Communicating Stories with Data	Data, Graphs & Tables	<p>SP.3, SP.4, EE.2, EE.5, F.3, F.4, F.5: Construct graphs of relationships between two variables (bivariate data), displaying frequencies and relative frequencies in a two-way table.</p> <ul style="list-style-type: none"> Use graphs with categorical data to help students describe events in their lives, looking at patterns in the graphs.
Communicating Stories with Data	Data Explorations	<p>SP.1, SP.2, SP.3, SP.4, EE.4, EE.5, F.1, F.2, F.3, F.4, F.5: Conduct data explorations, such as the consideration of seafloor spreading, involving large data sets and numbers expressed in scientific notation, including integer exponents for large and small numbers using technology.</p> <ul style="list-style-type: none"> Identify a large dataset and discuss the information it contains Identify what rows and columns represent in a spreadsheet
Exploring Changing Quantities	Linear Equations	<p>EE.5, EE.7, EE.8, F.2, F.4, F.5: Analyze slope and intercepts and solve linear equations including pairs of simultaneous linear equations through graphing and tables and using technology.</p>
Exploring Changing Quantities	Multiple Representations of Functions	<p>EE.5, EE.6, EE.7: Move between different representations of linear functions (i.e., equation, graph, table, and context), sketch and analyze graphs, use similar triangles to visualize slope and rate of change with equations containing rational number coefficients.</p>
Exploring Changing Quantities	Slopes & Intercepts	<p>EE.5, SP.1, SP.2, SP.3: Construct graphs using bivariate data, comparing the meaning of parallel and non-parallel slopes with the same or different y-intercepts using technology.</p>
Taking Wholes Apart, Putting Parts Together & Discovering Shape and Space	Cylindrical Investigations	<p>G.9, G.6, G.7, G.8, NS.1, NS.2: Solve real world problems with cylinders, cones, and spheres. Connect volume and surface area solutions to the structure of the figures themselves (e.g., why and how is the area of a circle formula used to find the volume of a cylinder?). Show visual proofs of these relationships, through modeling, building, and using computer software.</p>

Content Connection	Big Idea	Grade 8 Standards
Taking Wholes Apart, Putting Parts Together & Discovering Shape and Space	Pythagorean Explorations	G.7, G.8, NS.1, NS.2, EE.1, EE.2: Conduct investigations in the coordinate plane with right triangles to show that the areas of the squares of each leg combine to create the square of the hypotenuse and name this as the Pythagorean Theorem. Using technology, use the Pythagorean Theorem to solve real world problems that include irrational numbers.
Taking Wholes Apart, Putting Parts Together	Big & Small Numbers	EE.1, EE.2, EE.3, EE.4, NS.1, NS.2: Use scientific notation to investigate problems that include measurements of very large and very small numbers. Develop number sense with integer exponents (e.g., $1/27 = 1/3^3 = 3^{-3}$).
Discovering Shape and Space	Shape, Number & Expressions	G.9, G.6, G.7, G.8, EE.1, EE.2, NS.1, NS.2: Compare shapes containing circular measures to prisms. Note that cubes and squares represent unit measures for volume and surface area. See and use the connections between integer exponents and area and volume.
Discovering Shape and Space	Transformational Geometry	G.1, G.2, G.3, G.4, G.5, G.6, G.7, G.8: Plot two dimensional figures on a coordinate plane, using geometry software, noting similarity when dilations are performed and the corresponding angle measures maintain congruence. Perform translations, rotations, and reflections and notice when shapes maintain congruence.

Chapter 9: Mathematics in High School, Grades Nine and Ten

This section focuses on years 9 and 10, as the 2021 *Mathematics Framework* sets out a few common pathways for grades 9 and 10, followed by a plethora of options for later years, as set out in Figure 9.1:

Figure 9.1: High School Pathways from the 2021 Mathematics Framework



Long description: Diagram indicates three pathways of courses indicating a variety of course offerings. The preparatory courses are indicated at the bottom of the diagram. Starting from the bottom up, these include: Investigating and Connecting 1, Integrated 1, and Algebra 1. These are followed by Investigating and Connecting 2, Integrated 2, and Geometry, respectively. The later course options include: MIC – Modeling with Functions, Statistics, Calculus with Trigonometry, Pre-Calculus, Integrated 3, Algebra II, MIC – Data Science, and Other.

In the first two years of high school, the big ideas of mathematics fall into three main areas, which are ideally taught together rather than as separate courses. When the content is truly integrated, for example, students learning the content of functions through exciting data explorations, or analyzing the design of buildings with great cultural significance by studying and representing patterns of shapes and their transformations and vectors, then the content of high school comes to life. At this time, students can be learning to develop models and to use technological tools, such as geometry software, data modeling tools, such as CODAP, and programs that provide conceptual insights into computation, such as Desmos and Wolfram-Alpha.

Algebra and Functions

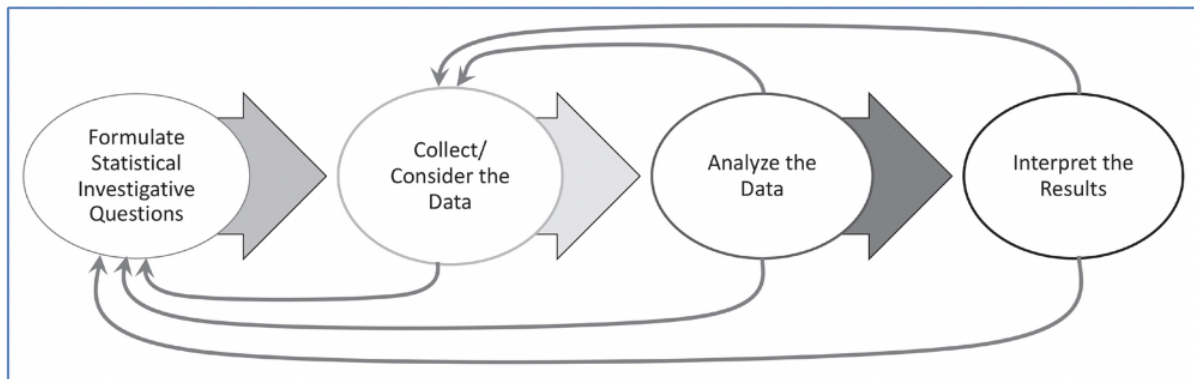
The number sense students developed in grades kindergarten through eight help students see the parallels between numbers (and how they interact) and functions, especially polynomials and rational functions. This area of mathematics should develop students' ability to recognize, represent, and solve problems involving relations among quantitative variables. Students can start to engage in the modelling of problems, using linear, quadratic, exponential, power, and polynomial functions. The investigations with this content, which fall under the area of "Exploring Changing Quantities" will help students to understand rates of change, growth, and decay functions, and many other topics important to solving problems in the world. As students investigate with the power of mathematical modeling, they will see the ways that they can use mathematics to make sense of the world and impact the future, two of the Drivers of Investigation. Many of the important algebraic topics can be taught through data investigations.

Data Literacy leading to Data Science

In the high school years, students develop their understanding of variability, learning how to measure and analyze variation. They can also be introduced to large and complex data sets and encouraged to ask their own questions of the data. Ideally, teachers will bring in data sets from their own communities, so that students can use mathematics to solve important problems that help their communities and develop their sense of mathematical agency (Berry et al., 2020). Randomization is an important high school understanding, leading to probability models and sample spaces. Technology plays an important role as it makes it possible for students to generate plots, regression functions, and correlation coefficients.

This is the beginning of moving from data literacy to the subject known as "data science" for students and a time when they can be using powerful technological tools to help visualize data distributions and analyze data. Data investigations, represented in Figure 9.2, involve many different areas of content, from within and outside mathematics, and students can be encouraged to report on their investigations, communicating their results with words, numbers, and data visualizations. The 2021 *Mathematics Framework* shares many ideas for data investigations that help students learn about social and racial injustices (such as redlining, voter suppression, wealth gap, and food insecurity), agriculture, the environment, healthcare, and other topics of importance for students in California.

Figure 9.2. GAISE Report 2020. (Franklin & Bargagliotti, 2020)



Visual and Geometric Reasoning

The third strand of the high school years with ideas from *Discovering Shape and Space & Taking Wholes Apart, Putting Parts Together*, is in visual and geometric reasoning. Students learn to construct and interpret mathematical models in visual and physical terms. They learn to describe patterns in shape, size, and location, representing patterns with drawings, coordinates, and vectors. Geometric ideas can be developed through experimentation and reasoning, while probabilistic ideas can be developed through geometric sample spaces.

Students who are English learners are encouraged to use their developing English and native language assets (e.g., cognates, morphological awareness) and draw on their prior knowledge. Teachers can examine text and tasks for key language forms and structures, and general as well as discipline-specific, high-utility academic vocabulary words linked to the big ideas and connections. Teachers can provide purposeful scaffolds and supports to engage EL students in sustained mathematical discourse in multiple contexts to build academic language and knowledge. Planned and “just-in-time” scaffolds and supports provide multiple entry points for meaning making and sharing of ideas in mathematical ways and include representations, expression starters and builders, and targeted and high-utility academic vocabulary and language structures (e.g., problems, explanations, arguments, descriptions, and connections). Teachers guide deconstruction and/or co-construction of problems, investigations, arguments, explanations, descriptions, and connections. Clear and precise expressions, as well as cohesive writing, support stronger communication of mathematical concepts and practices.

The following interview highlights an educator who is using digital tools to help students in this grade span express their thinking and collaborate with peers on problems, as well as to provide feedback and support as necessary.

Voices from the Field: Kristan Morales | Chaparral High School | Temecula, CA

With 25 years of experience in the classroom, math teacher Kristan Morales focuses on engagement, technology integration, and social-emotional learning. She continues to reevaluate and redesign her math instruction, pushing the envelope with what’s possible in math class in order to better serve her diverse learners.

Morales—who teaches geometry and pre-calculus and serves as a technology coach at Chaparral High School in the Temecula Valley Unified School District—has experienced high levels of student engagement, creativity, and learning outcomes.

How have you overcome the challenges of remote instruction, especially balancing synchronous and asynchronous learning experiences for your students?

For me, whether it's in-person or virtual, my dominant focus is always about student connections and connectivity. Granted, this has proven to be more challenging online. So, I knew right away that I would have to be intentional in creating opportunities to connect in virtual settings.

I created something I call "What's Up Weekly." It starts as a blank slideshow where every student creates a slide asynchronously. The students post their submission each week on a class Padlet [collaborative digital presentation software]. So, when I'm online with them, I refer to these slideshows and ask them to share. This is about creating community. I've learned so much about them, and they have, in turn, learned so much about one another.

Students buy into the relationship and the connection first, then into the math. One has to create and facilitate situations to hear students' voices. We have 90-minute classes, and there is no way that we are only going to listen to me talking. Some are reluctant, but I continue to create more ways for students to participate. We have such great tools now—all available to create these different avenues for students to be heard and for teachers to learn from students.

Even our traditional way of sharing our daily agenda has to be rethought. I now use a three-part Hyperdoc: explore, explain, and apply. Everyone can access this—students, parents, special education staff, and counselors.

In what ways are you combining analog and digital tools for distance learning?

There is no way around doing analog work in math. Math teachers have traditionally used notebooks to document student work and check for understanding or mastery. Well, we now can use very dynamic digital notebooks.

Math is built for analog and digital to co-exist and complement one another. For example, we solve problems on paper but can take pictures of them to document. I have my students work in teams and jigsaw the problems. They share with their teams using Jamboards [collaborative digital whiteboards] or Google Slides [collaborative digital presentation software].

There are so many ways to deploy digital tools with analog experiences. For example, while recently learning about radius and 2π , I asked my students to find a cylinder at home. It could be anything from lip gloss to a paper towel roll, coffee can, or skateboard wheel. As another example, we recently used household items and photography to study volume and students shared their work on Padlet. I have had success when my students do analog activities but then show their work digitally.

How do these digital tools used during remote instruction enhance what may be done in a traditional classroom setting?

For asynchronous times, I might focus on additional support. I may add a video of me thinking out loud while demonstrating something or working out a math problem. Many teachers will not make a mistake in their problem solving when creating a video. That will leave kids thinking that math is perfect, not messy. When in reality, we need to demonstrate the mistakes. When you make a mistake, neuroscience shows we are learning.

The tools available now can really improve the experience for learners. As an example, I use Desmos for graphing. This amazing free tool allows us to go much further and faster. Recently, we did a lesson using pendulums, and we collected the data in class using pendulums of different lengths. We then used our class data to curve fit the correct function using Desmos and our knowledge of functions. Once we found the best function to match our data, we made a prediction as to how much time it would take for a giant pendulum to swing, and our function from Desmos was very accurate.

We need to see end results more quickly and be able to visualize the learning. My students don't need to get bogged down in hand graphing. Geogebra is another free tool that is particularly useful for angles and relationships. Like Desmos, students are able to experience the dynamic properties in math. These tools have allowed students to see math in new and powerful ways. Technology doesn't replace analog but rather enhances the understanding and power of discovery for my students.

I really appreciate incorporating high quality tools that are free and available to all. I also really see value in tools that support all learners. For example, I think tools, such as Google Apps and Pear Deck [a formative assessment platform], are great because of the immersive reader or translation capabilities, making text and images larger and more.

I think tools that provide teachers with high-quality and real-time data and feedback on their students are important, too. Nearpod is a great formative assessment platform that provides teachers insight into what their students are thinking about, learning, or needing is useful. A recent Nearpod focused on special right triangles. Teachers know how to create content, but we need to continue to develop ways for all learners to engage in the content. We're not spending a lot of money on paper and copies any longer, so let's keep investing in technology and our students.

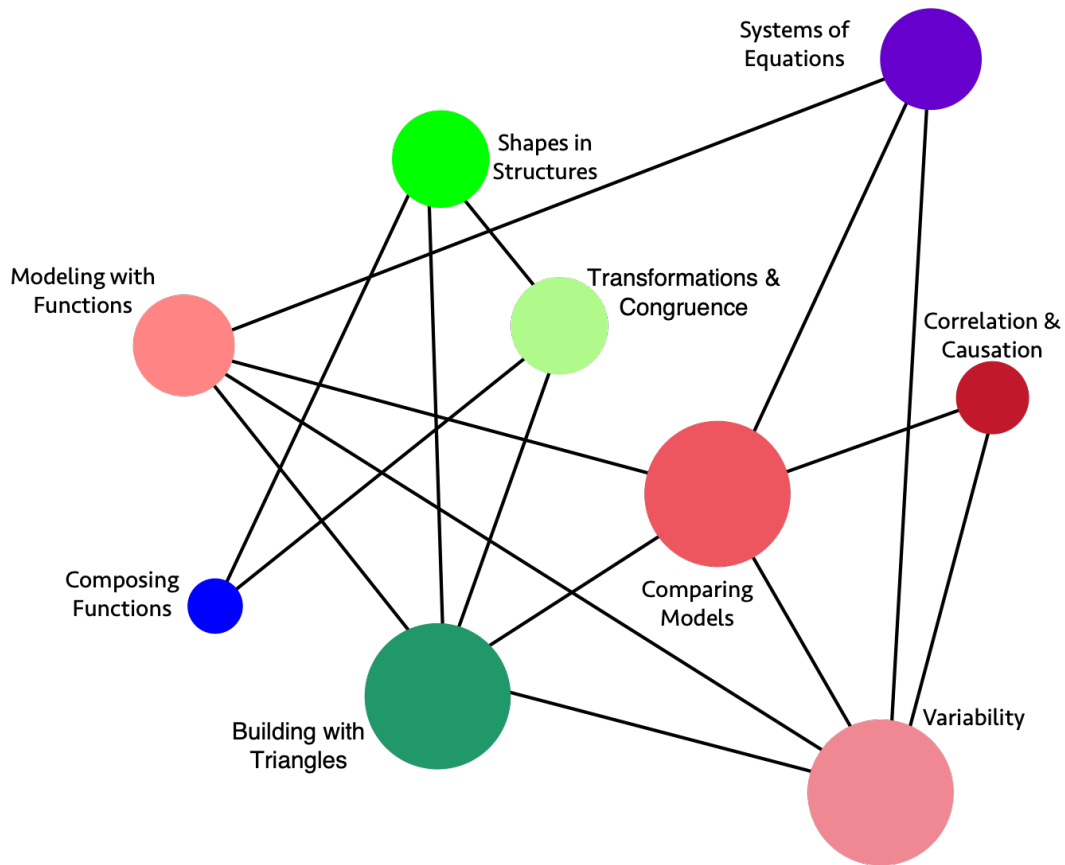
Figure 9.3. A Progression Chart of Big Ideas through Integrated 1 & 2

Content Connections	Big Ideas: Integrated 1	Big Ideas: Integrated 2
Communicating Stories with Data	Modeling with functions	The shape of distributions
Communicating Stories with Data	Comparing models	Geospatial data
Communicating Stories with Data	Variability	Probability modeling
Communicating Stories with Data	Correlation & causation	Experimental models and functions
Exploring Changing Quantities	Modeling with functions	The shape of distributions
Exploring Changing Quantities	Comparing models	Equations to predict & model

Content Connections	Big Ideas: Integrated 1	Big Ideas: Integrated 2
Exploring Changing Quantities	Variability	Experimental models & functions
Exploring Changing Quantities	Systems of equations	Transformation & similarity
Taking Wholes Apart, Putting Parts Together	Systems of equations	Functions in the world
Taking Wholes Apart, Putting Parts Together	Composing functions	Polynomial identities
Taking Wholes Apart, Putting Parts Together	Shapes in structures	Function representations
Taking Wholes Apart, Putting Parts Together	Building with triangles	n/a
Discovering shape and space	Shapes in structures	Circle relationships
Discovering shape and space	Building with triangles	Trig functions
Discovering shape and space	Transformations & congruence	Transformation & similarity

Critical Areas of Instructional Focus

Figure 9.4. High School Integrated 1 Big Ideas



Long description: The graphic illustrates the connections and relationships of some high school integrated mathematics concepts. Direct connections include:

- Systems of Equations directly connects to: Variability, Comparing Models, Modeling with Functions
- Correlation & Causation directly connects to: Variability, Comparing Models
- Variability directly connects to: Correlation & Causation, Comparing Models, Systems of Equations, Modeling with Functions, Building with Triangles
- Building with Triangles directly connects to: Variability, Comparing Models, Transformations & Congruence, Shapes in Structures, Modeling with Functions
- Composing Functions directly connects to: Transformations & Congruence, Shapes in Structures

- Modeling with Functions directly connects to: Building with Triangles, Variability, Comparing Models, Systems of Equations
- Shapes in Structures directly connects to: Transformations & Congruence, Building with Triangles, Composing Functions
- Transformations & Congruence directly connects to: Building with Triangles, Composing Functions, Shapes in Structures
- Comparing Models directly connects to: Correlation & Causation, Variability, Building with Triangles, Modeling with Functions, Systems of Equations

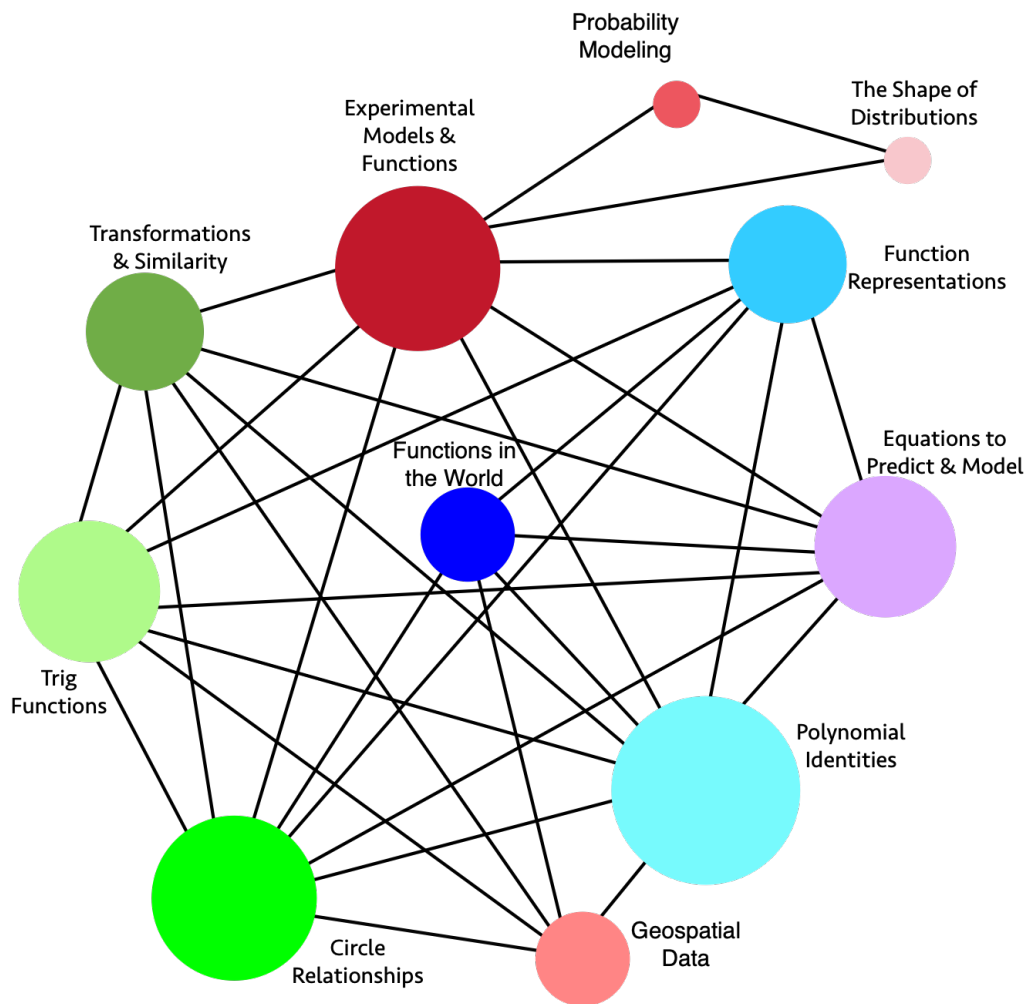
Figure 9.4a. High School Integrated 1 Content Connections, Big Ideas, and Standards

Content Connection	Big Idea	Integrated 1 Standards
Communicating Stories with Data & Exploring Changing Quantities	Modeling with Functions	N-Q.1, N-Q.2, N-Q.3, A-CED.2, F-BF.1, F-IF.1, F-IF.2, F-IF.4, F-LE.5, S-ID.7, A-CED.1, A-CED.2, A-CED.3, A-SSE.1: Build functions that model relationships between two quantities, including examples with inequalities; using units and different representations. Describe and interpret the relationships modeled using visuals, tables, and graphs.
Communicating Stories with Data & Exploring Changing Quantities	Comparing Models	F-LE.1, F-LE.2, F-LE.3, F-IF.4, F-BF.1, F-LE.5, S-ID.7, S-ID.8, A-CED.1, A-CED.2, A-CED.3, A-SSE.1: Construct, interpret, and compare linear, quadratic, and exponential models of real data, and use them to describe and interpret the relationships between two variables, including inequalities. Interpret the slope and constant terms of linear models, and use technology to compute and interpret the correlation coefficient of a linear fit.
Communicating Stories with Data & Exploring Changing Quantities	Variability	S-ID.5, S-ID.6, S-ID.7, S-ID.1, S-ID.2, S-ID.3, S-ID.4, A-SSE.1: Summarize, represent, and interpret data. For quantitative data, use a scatter plot and describe how the variables are related. Summarize categorical data in two-way frequency tables and interpret the relative frequencies.
Communicating Stories with Data	Correlation & Causation	S-ID.9, S-ID.8, S-ID.7: Explore data that highlights the difference between correlation and causation. Understand and use correlation coefficients, where appropriate. (see resource section for classroom examples).

Content Connection	Big Idea	Integrated 1 Standards
Exploring Changing Quantities & Taking Wholes Apart, Putting Parts Together	Systems of Equations	A-REI.1, A-REI.3, A-REI.4, A-REI.5, A-REI.6, A-REI.7, A-REI.10, A-REI.11, A-REI.12, NQ.1, A-SEE.1: Students investigate real situations that include data for which systems of 1 or 2 equations or inequalities are helpful, paying attention to units. Investigations include linear, quadratic, and absolute value. Students use technology tools strategically to find their solutions and approximate solutions, constructing viable arguments, interpreting the meaning of the results, and communicating them in multidimensional ways.
Taking Wholes Apart, Putting Parts Together	Composing Functions	F-BF.3, F-BF.2, F-IF.3: Build and explore new functions that are made from existing functions, and explore graphs of the related functions using technology. Recognize sequences are functions and are defined recursively.
Taking Wholes Apart, Putting Parts Together & Discovering Shape and Space	Shapes in Structures	G-CO.6, C-CO.7, C-CO.8, G-GPE.4, G-GPE.5, G.GPE.7, F.BF.3: Perform investigations that involve building triangles and quadrilaterals, considering how the rigidity of triangles and non-rigidity of quadrilaterals influences the design of structures and devices. Study the changes in coordinates and express the changes algebraically.
Taking Wholes Apart, Putting Parts Together & Discovering Shape and Space	Building with Triangles	G-GPE.4, G-GPE.5, G-GPE.6, GPE.7, F-LE.1, F-LE.2, A-CED.2: Investigate with geometric figures, constructing figures in the plane, relating the distance formula to the Pythagorean Theorem, noticing how areas and perimeters of polygons change as the coordinates change. Build with triangles and quadrilaterals, noticing positions and movement, and creating equations that model the changing edges using technology.
Discovering Shape and Space	Transformations & Congruence	G-CO.1, G-CO.2, G-CO.3, G-CO.4, G-CO.5, G-CO.12, G-CO.13, G-GPE.4, G-GPE.5, G.GPE.7, F-BF.3: Explore congruence of triangles, including quadrilaterals built from triangles, through geometric constructions. Investigate transformations in the plane. Use geometry software to study transformations, developing definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, and parallel lines. Express translations algebraically.

Critical Areas of Instructional Focus

Figure 9.5. High School Integrated 2 Big Ideas



Long description: The graphic illustrates the connections and relationships of some high school integrated mathematics concepts. Direct connections include:

- Function Representations directly connects to: Equations to Predict & Model, Polynomial Identities, Circle Relationships, Functions in the World, Trig Functions, Experimental Models & Functions
- Equations to Predict & Model directly connects to: Polynomial Identities, Circle Relationships, Trig Functions, Functions in the World, Transformations & Similarity, Experimental Models & Functions, Function Representations
- Polynomial Identities directly connects to: Geospatial Data, Circle Relationships, Trig Functions, Transformations & Similarity, Functions in the World, Experimental Models & Functions, Function Representations, Equations to Predict & Model
- Geospatial Data directly connects to: Polynomial Identities, Functions in the World, Transformations & Similarity, Trig Functions, Circle Relationships

- Circle Relationships directly connects to: Geospatial Data, Polynomial Identities, Trig Functions, Transformations & Similarity, Functions in the World, Experimental Models & Functions, Function Representations, Equations to Predict & Model
- Trig Functions directly connects to: Geospatial Data, Circle Relationships, Polynomial Identities, Transformations & Similarity, Experimental Models & Functions, Function Representations, Equations to Predict & Model
- Transformations & Similarities directly connects to: Geospatial Data, Circle Relationships, Trig Functions, Polynomial Identities, Experimental Models & Functions, Equations to Predict & Model
- Experimental Models & Functions directly connects to: Circle Relationships, Trig Functions, Transformations & Similarity, Polynomial Identities, Function Representations, Equations to Predict & Model, The Shape of Distributions, Probability Modeling
- Probability Modeling directly connects to: The Shape of Distributions, Experimental Models & Functions
- The Shape of Distributions directly connects to: Probability Modeling, Experimental Models & Functions
- Functions in the world directly connects to: Functions Representations, Equations to Predict & Model, Polynomial Identities, Geospatial Data, Circle Relationships

Figure 9.5a. High School Integrated 2 Content Connections, Big Ideas, and Standards

Content Connection	Big Idea	Integrated 2 Standards
Communicating Stories with Data	Probability Modeling	S.CP.1, S.CP.2, S.CP.3, S.CP.4, S.CP.5, S-IC.1, S-IC.2, S-IC.3, S.MD.6, S.MD.7: Explore and compare independent and conditional probabilities, interpreting the output in terms of the model. Construct and interpret two-way frequency tables of data as a sample space to determine if the events are independent, and use the data to approximate conditional probabilities. Examples of topics include product and medical testing, and player statistics in sports.
Communicating Stories with Data	The shape of distributions	S-IC.1, S-IC.2, S-IC.3, S-ID.1, S-ID.2, S-ID.3, S-MD.1, S-MD.2: Consider the shape of data distributions to decide on ways to compare the center and spread of data. Use simulation models to generate data, and decide if the model produces consistent results.

Content Connection	Big Idea	Integrated 2 Standards
Communicating Stories with Data & Exploring Changing Quantities	Experimental Models & Functions	S-ID.1, S-ID.2, S-ID.3, S-ID.6, S-ID.7, S-IC.1, S-IC.2, S-IC.3, A-CED.1, A-REI.1, A-REI.4, F-IF.2, F-IF.3, F-IF.4, F-BF.1, F-LE.1, F-TF.2, A-APR.1: Conduct surveys, experiments, and observational studies - drawing conclusions and making inferences. Compare different data sources and what may be most appropriate for the situation. Create and interpret functions that describe the relationships, interpreting slope and the constant term when linear models are used. Include quadratic and exponential models when appropriate, and understand the meaning of outliers.
Communicating Stories with Data	Geospatial Data	G-MG.1, G-MG.2, G-MG.3, F-LE.6, G-GPE.4, G-GPE.6, G-SRT.5, G-CO.1, G-CO.2, G-CO.12, G-C.2, G-C.5: Explore geospatial data that represent either locations (e.g., maps) or objects (e.g., patterns of people's faces, road objects for driverless cars) and connect to geometric equations and properties of common shapes. Demonstrate how a computer can measure the distance between two points using geometry and then account for constraints (e.g., distance and then roads for directions) and multiple points with triangulation. Model what shapes and geometric relationships are most appropriate for different situations.
Exploring Changing Quantities	Equations to Predict & Model	A-CED.1, A-CED.2, A-REI.4, A-REI.1, A-REI.2, A-REI.3, F-IF.4, F-IF.5, F-IF.6, F-BF.1, F-BF.3, A-APR.1: Model relationships that include creating equations or inequalities, including linear, quadratic, and absolute value. Use the equations or inequalities to make sense of the world or to make predictions, understanding that solving equations is a process of reasoning. Make sense of the real situation, using multiple representations, such as graphs, tables, and equations.
Taking Wholes Apart, Putting Parts Together	Functions in the World	F-LE.3, F-LE.6, F-IF.9, N-RN.1, N-RN.2, A-SSE.1, A-SSE.2: Apply quadratic functions to the physical world, such as motion of an object under the force of gravity. Produce equivalent forms of the functions to reveal zeros, max and min, and intercepts. Investigate how functions increase and decrease, and compare the rates of increase or decrease to linear and exponential functions.
Taking Wholes Apart, Putting Parts Together	Polynomial Identities	A-SSE.1, A-SSE.2, A-APR.1, A-APR.3, A-APR.4, G-GMD.2, G-MG.1, S-IC.1, S-MD.2: Prove polynomial identities, and use them to describe numerical relationships, using a computer algebra system to rewrite polynomials. Use the binomial theorem to solve problems, appreciating the connections with Pascal's triangle.

Content Connection	Big Idea	Integrated 2 Standards
Taking Wholes Apart, Putting Parts Together	Functions Representations	F-IF.4, F-IF.5, F-IF.6, F-IF.7, F-IF.8, F-IF.9, N-RN.1, N-RN.2, F-LE.3, A-APR.1: Interpret functions representing real world applications in terms of the data understanding key features of graphs, tables, domain, and range. Compare properties of two functions each represented in different ways (algebraically, graphically, numerically, in tables or by written/verbal descriptions).
Discovering Shape and Space & Exploring Changing Quantities	Transformations & Similarity	G-SRT.1, G-SRT.2, G-SRT.3, , A-CED.2, G-GPE.4, F-BF.3, F-IF.4, A-APR.1: Explore similarity and congruence in terms of transformations, noticing the changes dilations have on figures and the effect of scale factors. Discover how coordinates can be used to describe translations, rotations, and reflections, and generalize findings to model the transformations using algebra.
Discovering Shape and Space	Circle Relationships	G-C.1, G-C.2, G-C.3, G-C.4, G-C.5, G-GPE.1, A-REI.7, A-APR.1, F-IF.9: Investigate the relationships of angles, radii, and chords in circles, including triangles and quadrilaterals that are inscribed and circumscribed. Explore arc lengths and areas of sectors using the coordinate plane. Relate the Pythagorean Theorem to the equation of the circle given the center and radius, and solve simple systems where a line intersects the circle.
Discovering Shape and Space	Trig Functions	G-TF.2, G-GPE.1, G-GMD.2, G-MG.1, A-APR.1: Model periodic phenomena with trigonometric functions. Translate between geometric descriptions and the equation for a conic section. Visualize relationships between 2-D and 3-D objects.

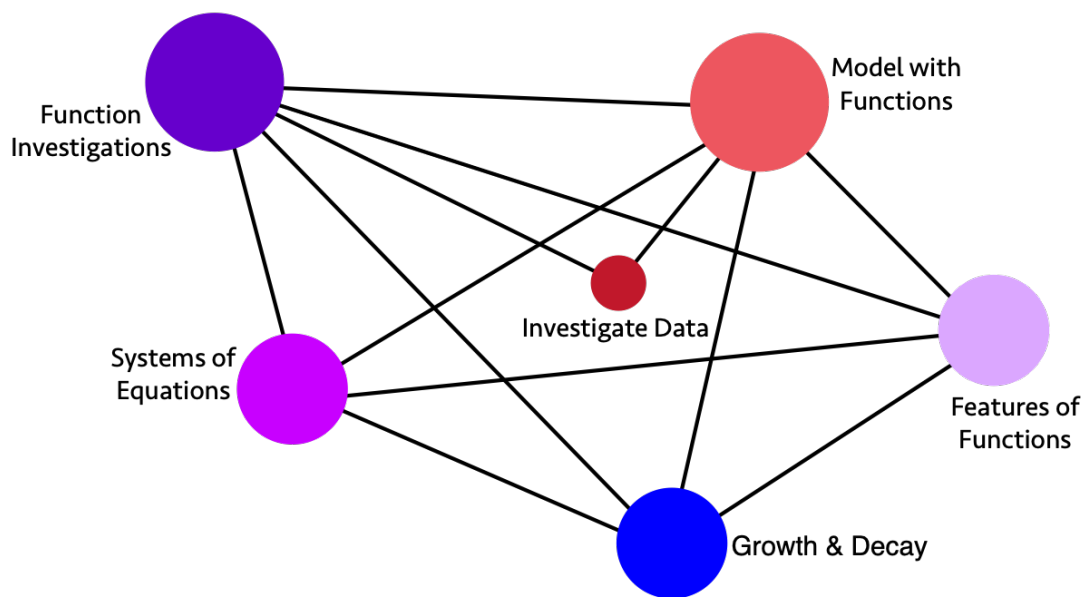
Figure 9.6. A Progression Chart of Big Ideas through Algebra and Geometry

Content Connections	Big Ideas: Algebra	Big Ideas: Geometry
Communicating Stories with Data	Investigate data	Fairness in data
Communicating Stories with Data	Model with functions	Geospatial data
Communicating Stories with Data	n/a	Probability modeling
Exploring Changing Quantities	Function investigations	Trig explorations

Content Connections	Big Ideas: Algebra	Big Ideas: Geometry
Exploring Changing Quantities	Systems of equations	Triangle congruence
Exploring Changing Quantities	Features of functions	Triangle problems
Exploring Changing Quantities	n/a	Circle relationships
Exploring Changing Quantities	n/a	Points & slopes
Taking Wholes Apart, Putting Parts Together	Growth & decay	Triangle congruence
Taking Wholes Apart, Putting Parts Together	n/a	Transformations
Discovering shape and space	Model with functions	Triangle congruence
Discovering shape and space	Investigate data	Transformations
Discovering shape and space	n/a	Circle relationships
Discovering shape and space	n/a	Geometric models

Critical Areas of Instructional Focus

Figure 9.7. High School Algebra Big Ideas



Long description: The graphic illustrates the connections and relationships of some high school algebra mathematics concepts. Direct connections include:

- Model with Functions directly connects to: Features of Functions, Growth & Decay, Investigate Data, Systems of Equations, Function Investigations
- Features of Functions directly connects to: Growth & Decay, Systems of Equations, Function Investigations, Model with Functions
- Growth & Decay directly connects to: Features of Functions, Model with Functions, Function Investigations, Systems of Equations
- Systems of Equations directly connects to: Growth & Decay, Features of Functions, Model with Functions, Function Investigations
- Function Investigations directly connects to: Model with Functions, Features of Functions, Growth & Decay, Investigate Data, Systems of Equations
- Investigate Data directly connects to: Model with Functions, Function Investigations

Figure 9.7a. High School Algebra Content Connections, Big Ideas, and Standards

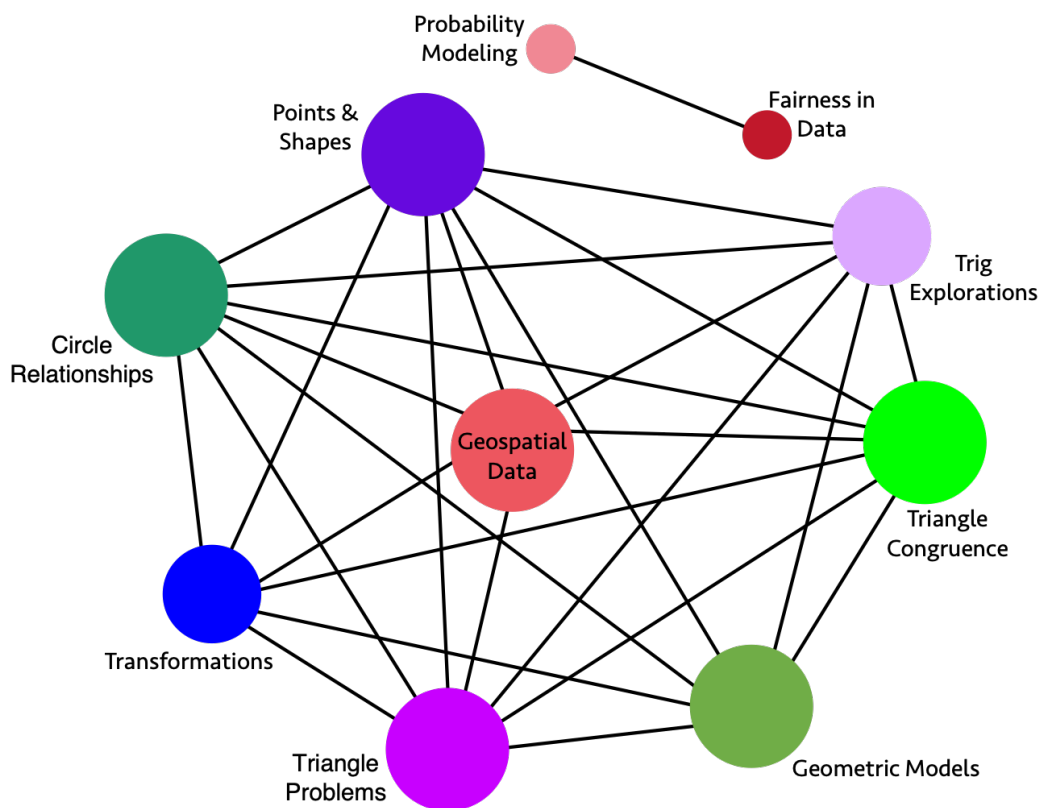
Content Connection	Big Idea	Algebra Standards
Communicating Stories with Data & Discovering Shape and Space	Investigate Data	<p>S-ID.1, S-ID.2, S-ID.3, S-ID.6: Represent data from two or more data sets with plots, dot plots, histograms, and box plots, comparing and analyzing the center and spread, using technology, and interpreting the results. Interpret and compare data distributions using center (median, mean) and spread (interquartile range, standard deviation) through the use of technology.</p> <ul style="list-style-type: none"> ● Students have opportunities to explore and research a topic of interest and meaning to them, using the statistical methods, tools, and representations. ● Have students consider how different, competing interpretations can be made from different audiences, histories, and perspectives. ● Allow students to develop follow-up questions to investigate, spurred by the original data set.
Communicating Stories with Data & Discovering Shape and Space	Model with Functions	<p>F-IF.1, F-IF.2, F-IF.4, F-IF.5, F-IF.6, F-IF.7, F-IF.8, F-IF.9, F-BF.1, F-BF.2, F-BF.4, F-LE.1, F-LE.2, S-ID.5, S-ID.6, S-ID.7, S-ID.8, S-ID.9: Investigate data sets by table and graph and using technology; fit and interpret functions** to model the data between two quantities. Interpret information from the functions, noticing key features* and symmetries. Develop understanding of the meaning of the function and how it represents the data that it is modeling; recognizing possible associations and trends in the data - including consideration of the correlation coefficients of linear models.</p> <ul style="list-style-type: none"> ● Students can disaggregate data by different characteristics of interest (populations for example), and compare slopes to examine questions of fairness and bias among groups. ● Students have opportunities to consider how to communicate relevant concerns to stakeholders and/or community members. ● Students can identify both extreme values (true outliers) and data errors, and how the inclusion or exclusion of these observations may change the function that would most appropriately model the data. <p>*intercepts, slope, increasing or decreasing, positive or negative ** functions include linear, quadratic and exponential</p>

Content Connection	Big Idea	Algebra Standards
Exploring Changing Quantities	Systems of Equations	<p>A-REI.1, A-REI.3, A-REI.4, A-REI.5, A-REI.6, A-REI.7, A-REI.10, A-REI.11, A-REI.12, NQ.1, A-SEE.1, F-LE.1, F-LE.2: Students investigate real situations that include data for which systems of 1 or 2 equations or inequalities are helpful, paying attention to units. Investigations include linear, quadratic, and absolute value. Students use technology tools strategically to find their solutions and approximate solutions, constructing viable arguments, interpreting the meaning of the results, and communicating them in multidimensional ways.</p>
Exploring Changing Quantities	Function investigations	<p>F-IF.1, F-IF.2, F-IF.4, F-IF.5, F-IF.6, F-IF.7, F-IF.8, F-IF.9, F-BF.1, F-BF.2, F-BF.4, S-ID.5, S-ID.6, S-ID.7, S-ID.8, S-ID.9, F-LE.1, F-LE.2: Students investigate data sets by table and graph and using technology; such as earthquake data in the region of the school; they fit and interpret functions to model the data between two quantities and consider the meaning of inverse relationships. Students interpret information from the functions, noticing key features* and symmetries. Students develop understanding of the meaning of the function and how it represents the data that it is modeling; they recognize possible associations and trends in the data - including consideration of the correlation coefficients of linear models.</p> <p>*one to one correspondence, intercepts, slope, increasing or decreasing, positive or negative</p>
Exploring Changing Quantities	Features of Functions	<p>A-SSE.3, F-IF.3, F-IF.4, F-LE.1, F-LE.2, F-LE.6: Students investigate changing situations that are modeled by quadratic and exponential forms of expressions and create equivalent expressions to reveal features* that help understand the meaning of the problem and situation being investigated. (driver of investigation 1, making sense of the world)</p> <p>Investigate patterns, such as the Fibonacci sequence and other mathematical patterns, that reveal recursive functions.</p> <p>*Factored form to reveal zeros of a quadratic function, standard form to reveal the y-intercept, vertex form to reveal a maximum or minimum.</p>

Content Connection	Big Idea	Algebra Standards
Taking Wholes Apart, Putting Parts Together	Growth & Decay	F-LE.1, F-LE.2, F-LE.3, F-LE.5, F-LE.6, F-BF.1, F-BF.2, F-BF.3, F-BF.4, F-IF.4, F-IF.5, F-IF.9, NQ.1, A-SEE.1: Investigate situations that involve linear, quadratic, and exponential models, and use these models to solve problems. Recognize linear functions grow by equal differences over equal intervals; exponential functions grow by equal factors over equal intervals, and functions grow or decay by a percentage rate per unit interval. Interpret the inverse of functions, and model the inverse in graphs, tables, and equations.

Critical Areas of Instructional Focus

Figure 9.8 High School Geometry Big Ideas



Long description: The graphic illustrates the connections and relationships of some high school geometry mathematics concepts. Direct connections include:

- Probability Modeling directly connects to: Fairness in Data
- Fairness in Data directly connects to: Probability Modeling
- Trig Explorations directly connects to: Triangle Congruence, Geometric Models, Triangle Problems, Geospatial Data, Circle Relationships, Points & Shapes
- Triangle Congruence directly connects to: Geometric Models, Triangle Problems, Transformations, Geospatial Data, Circle Relationships, Points & Shapes, Trig Explorations
- Geometric Models directly connects to: Triangle Problems, Transformations, Circle Relationships, Points & Shapes, Trig Explorations, Triangle Congruence
- Triangle Problems directly connects to: Geometric Models, Triangle Congruence, Transformations, Geospatial Data, Circle Relationships, Points & Shapes, Trig Explorations
- Transformations directly connects to: Geometric Models, Triangle Problems, Triangle Congruence, Geospatial Data, Circle Relationships, Points & Shapes
- Circle Relationships directly connects to: Geometric Models, Triangle Problems, Transformations, Geospatial Data, Triangle Congruence, Points & Shapes, Trig Explorations
- Points & Shapes directly connects to: Geometric Models, Triangle Problems, Transformations, Geospatial Data, Circle Relationships, Triangle Congruence, Trig Explorations
- Geospatial Data: Triangle Problems, Transformations, Triangle Congruence, Circle Relationships, Points & Shapes, Trig Explorations

Figure 9.8a. High School Geometry Content Connections, Big Ideas, and Standards

Content Connection	Big Idea	Geometry Standards
Communicating Stories with Data	Probability Modeling	S-CP.1, S-CP.2, S-CP.3, S-CP.4, S-CP.5, S-IC.1, S-IC.2, S-IC.3, S-MD.6, S-MD.7: Explore and compare independent and conditional probabilities, interpreting the output in terms of the model. Construct and interpret two-way frequency tables of data as a sample space to determine if the events are independent and use the data to approximate conditional probabilities. Examples of topics include product and medical testing, and player statistics in sports.

Content Connection	Big Idea	Geometry Standards
Communicating Stories with Data	Fairness in Data	S-MD.6, S-MD.7: Determine fairness and make decisions based on evaluation of outcomes. Allow students to explore fairness by researching topics of interest, analyzing data from two-way tables. Provide opportunities for students to make meaningful inference, and communicate their findings to community or other stakeholders.
Communicating Stories with Data	Geospatial Data	G-MG.1, G-MG.2, G-MG.3, F-LE.6, G-GPE.4, G-GPE.6, G-SRT.5, G-CO.1, G-CO.2, G-CO.12, G-C.2, G-C.5: Explore geospatial data that represent either locations (e.g., maps) or objects (e.g., patterns of people’s faces, road objects for driverless cars), and connect to geometric equations and properties of common shapes. Demonstrate how a computer can measure the distance between two points using geometry, and then account for constraints (e.g., distance and then roads for directions) and multiple points with triangulation. Model what shapes and geometric relationships are most appropriate for different situations.
Exploring Changing Quantities	Trig Explorations	G-SRT.1, G-SRT.2, G-SRT.3, G-SRT.5, G-SRT.9, G-SRT.10, G-SRT.11, GPE.7, G-C.2, G-C.4: Investigate properties of right triangle similarity and congruence and the relationships between sine, cosine, and tangent; exploring the relationship between sine and cosine of complementary angles, and apply that knowledge to problem solving situations. Students recognize the role similarity plays in establishing trigonometric functions, and they use trigonometric functions to investigate situations. Using dynamic geometric software students investigate similarity and trigonometric identities to derive the Laws of Sines and Cosines and use the laws to solve problems.
Exploring Changing Quantities	Triangle Problems	G-SRT.4, G-SRT.5, G-SRT.6, G-SRT.8, G-C.2, G-C.4, G-CO.12: Understand and use congruence and similarity when solving problems involving triangles, including trigonometric ratios. Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems using dynamic geometric software.

Content Connection	Big Idea	Geometry Standards
Exploring Changing Quantities	Points & Shapes	<p>G-GPE.1, G-GPE.2, G-GPE.4, G-GPE.5, G-GPE.6, G-GPE.7, G-CO.1, G-CO.12, G-C.2, G-C.4: Solve problems involving geometric shapes in the coordinate plane using dynamic geometric software to apply the distance formula, Pythagorean Theorem, slope, and similarity rules in solving problems.</p> <ul style="list-style-type: none"> • Investigate equations of circles and how coefficients in the equations correspond to the location and radius of the circles. • Find areas and perimeters of triangles and rectangles in the coordinate plane.
Taking Wholes Apart, Putting Parts Together & Discovering Shape and Space	Transformations	<p>G-CO.1, G-CO.3, G-CO.4, G-CO.5, G-CO.12: Understand rotations, reflections, and translations of regular polygons, quadrilaterals, angles, circles, and line segments. Identify transformations, through investigation, that move a figure back onto itself, using that process to prove congruence.</p>
Discovering Shape and Space & Exploring Changing Quantities & Taking Wholes Apart, Putting Parts Together	Triangle Congruence	<p>G-CO.1, G-CO.2, G-CO.7, G-CO.8, G-CO.9, G-CO.10, G-CO.11, G-CO.12, G-CO.13, G-SRT.5: Investigate triangles and their congruence over rigid transformations verifying findings using triangle congruence theorems (ASA, SSS, SAS, AAS, and HL) and other geometric properties, including vertical angles, angles created by transversals across parallel lines, and bisectors.</p>
Discovering Shape and Space & Exploring Changing Quantities	Circle Relationships	<p>G-C.1, G-C.2, G-C.3, G-C.4, G-CO.1, G-CO.12, G-CO.13, G-GPE.1: Investigate similarity in circles and relationships between angle measures and segments, including inscribed angles, radii, chords, central angles, inscribed angles, circumscribed angles, and tangent lines using dynamic geometric software.</p>

Content Connection	Big Idea	Geometry Standards
Discovering Shape and Space	Geometric Models	<p>G-GMD.1, G-GMD.3, G-GMD.4, G-GMD.5, G-MG.1, G-MG.12, G-MG.13, SRT.5, G-CO.12, G-C.2, G-C.4: Apply geometric concepts in modeling situations to solve design problems using dynamic geometric software.</p> <ul style="list-style-type: none"> ● Investigate 3-D shapes and their cross sections. ● Use volume, area, circumference, and perimeter formulas. ● Understand and apply Cavalieri’s principle. ● Investigate and apply scale factors for length, area, and volume.

Note: Digital tools and resources to support the implementation of the strategies and considerations identified in this section are included in the Appendices. Please also note that digital tools referenced in Appendix B include free and premium options, and their inclusion in the guidance are largely derived from interviews with California educators. LEAs exercise local control when selecting digital tools and resources. Resources and digital tools included in the guide should not be considered endorsements by the CDE.

