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What is Mathematical Beauty?

Teaching through Big Ideas and Connections

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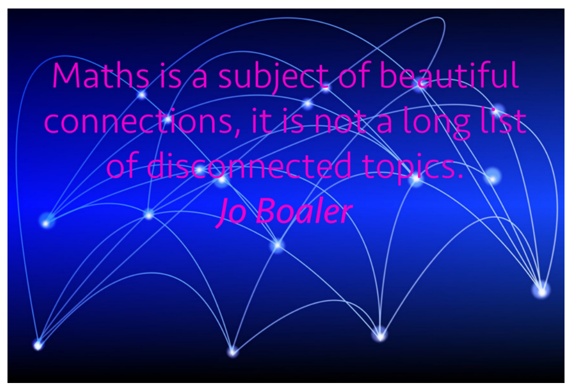
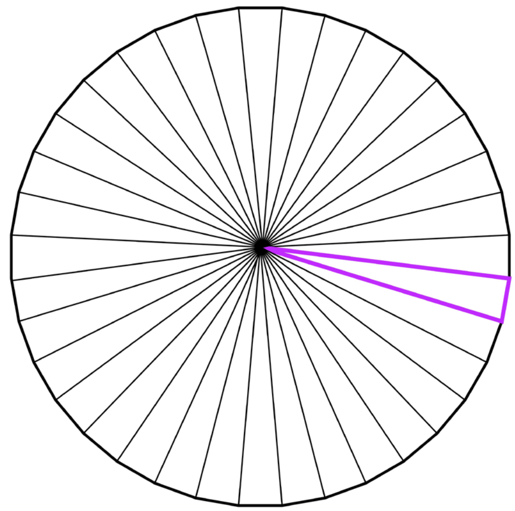
Mathematics is a beautiful subject. Ask mathematicians and others what they love about the subject and they will talk about the amazing connections that thread through the terrain, unifying the different ideas. There are not many facts or methods to remember in mathematics but there are a few really big and important ideas that are connected to each other and that infuse the subject. Yet when we ask students what they think math is, most will say that it is a lot of different rules and methods. This is really unfortunate as students who believe mathematics is a set of methods to be remembered are the lowest achieving students, worldwide, as revealed by PISA data (Boaler & Zoido, 2016). So, why do so few students, or teachers, see mathematics as a set of rich ideas and connections? One of the reasons is that teachers are given sets of standards to teach and no matter how good the standard writers are, they all cut mathematics up into small pieces and give teachers small atomized content areas – usually a set of methods – to teach. The connections disappear – teachers cannot see them and they are lost from students' learning pathways. Instead, teachers see the lists of content – often 100 or more methods in a year – and work systematically through them. This often leads teachers to skim through content quickly as when mathematics is disconnected and offered in small sections there is a lot to get through in any year. The connections between ideas are invisible and students do not develop one of the most important insights into mathematics that they can develop - that mathematics is a set of connected, big ideas.



An important step in seeing mathematical connections is knowing the big ideas in mathematics. I (JB) have systematically studied different school approaches in different countries, for many years. In two longitudinal, experimental studies of students learning mathematics in middle and high school, one in the UK and one in the US, I followed students through different teaching approaches. In both cases the students who achieved at higher levels and who enjoyed mathematics more had learned mathematics through more conceptual approaches and the teachers had planned the teaching around big ideas. In the UK, the successful teachers used a project based approach. They decided on the big ideas in the different years of mathematics and then chose long, open-ended projects through which students would encounter a need for different mathematical methods. I followed students from when they were aged 13 to when they were 16, collecting multiple forms of data. One of the projects students worked on was called 'Interpreting the World'. Students collected different forms of data about the world, that they chose, and analyzed. Another project was called '36 Fences' – students were told that a farmer had 36 fences and wanted to maximize the area of a fence made

> ***Fence Task***
> *A farmer wants to make the largest enclosure she can out of 36, 1-meter long, pieces of fencing. What is the biggest area of the fence possible?*

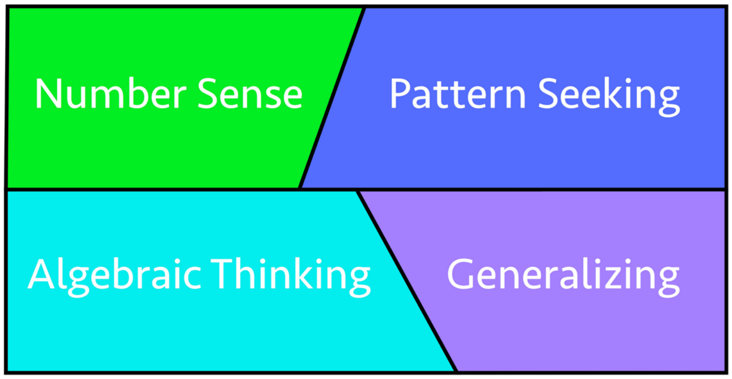
from them. Students were not told methods in advance of needing them – for example, when students worked on the 36 fences project, some of them made a 36-sided fence and realized they could find the area by making 36 triangles with a base of 1. To find the area of each triangle the students needed trigonometry and so teachers taught the students trigonometric methods when they encountered a need to use them, to solve the problem they were working on. At the end of three years of this approach the students scored at significantly higher levels on national mathematics exams, than students who worked systematically through method after method in textbook questions. Studies of the students six years later showed that students who learned through a project based approach ended up in more professional jobs, than those who had learned traditionally, which they attributed to the ways they had learned mathematics (see Boaler & Selling, 2017). https://www.youcubed.org/resources/psychological-imprisonment-intellectual-freedom-longitudinal-study-contrasting-school-mathematics-approaches-impact-adults-lives/



36-gon

In the US I studied students moving through three different high schools, from aged 14 to 18. In the most successful approach the teachers had again planned the teaching by working out big ideas for each year and then organized units and questions around those. The approach was conceptual but not project based – students worked on big ideas such as "What is a function?" and encountered different questions which they discussed in groups. The approach highlighted the connections between ideas, through methods such as color coding. Both approaches are described in more detail elsewhere (see https://www.youcubed.org/resource/short-impact-papers/), what is important to note here is that in both teaching cases teachers had worked out the big ideas in mathematics and then planned activities around them.

A third case of teaching that highlights the importance of big ideas is one on which two of the authors of this paper were the teachers – myself and Cathy Williams. We joined three other teachers in running and teaching an 18-day summer camp for middle school students at Stanford. The students spent three hours in the morning in mathematics lessons, and in the afternoons engaged in activities around campus. When the 81 students came to us, they all told interviewers that they were "not a math person", despite being from a wide range of achievement levels. The students had all taken an algebra test in their school district before coming to the camp, and we gave them the same test to take at the end of the camp, 18 days later. The students had improved by an average of 50%, which is equivalent to 2.4 years of school. The improvement of the students was due to many factors, including the infusion of brain science knowledge into our teaching. We told the students frequently that there was no such thing as a math person, brains grow and change, times of mistakes and challenge are the most important for brain growth, and that mathematics is about depth not speed. We also pointed out that when we encounter mathematics in different forms – visually, in words, numbers, algorithms, tables, graphs and other forms this encourages brain connections and brain growth. (The different findings from brain science that infused our teaching are all explained on youcubed.org.) Importantly we had planned the activities that we chose for the 18 lessons around big ideas, and students were encouraged to make connections between the ideas. The students were mainly finishing 6th grade and we chose to teach around these big ideas:



Big ideas from the youcubed summer math camp, 2015

We then chose 24 rich and engaging activities that were focused on those big ideas, that gave students many opportunities to see connections between ideas that were conceptual and engaging. (Examples of the types of activities we chose can be seen here: https://www.youcubed.org/tasks/) As students worked on the activities focused on big ideas, students encountered a need for many of the smaller methods and we taught those to students within the activities. This approach of teaching to big ideas and teaching smaller ideas when they arise, has the advantage of students always wanting to learn the smaller methods as they have a need for them to help solve problems. As an example of this, one of our activities that we chose to teach number sense was called Four 4's. The students were asked to try and find every number from 1 to 20 using exactly four 4's and any operation. All the numbers from 1 to 20 can be derived with four 4's but some of them need the operation factorial (!). With factorial you can obtain 24 as 4! is 4 x 3 x 2 x 1. We did not teach factorial at the beginning of the activity we waited for the students to realize they could not find some numbers without it. This made the learning of factorial very exciting for them and many thought it was very cool!

Factorial

$1! = 1 = 1$

$2! = 2 \times 1 = 2$

$3! = 3 \times 2 \times 1 = 6$

$4! = 4 \times 3 \times 2 \times 1 = 24$

This teaching of methods to students when they encounter a need for them has been shown in research to be the most effective method for teaching mathematics (Schwartz & Bransford, 1998). Students can learn through activities focused upon big ideas and when they encounter a need for a new method they learn it within the activity. When students do this their brains are primed to learn the new method as they are curious and they need the method. When teaching through the big ideas in mathematics most of the smaller ideas and methods naturally arise and students can learn them in a meaningful purposeful way. The ideas that never come up are probably not very important to learn!

The school in England that I studied taught through long open projects for three years and in the last few weeks before the national exam, went through examination papers with the students, teaching any methods they had not encountered during their projects. The school in the US taught most of the standard methods and did not worry about those that did not occur naturally in their units of work. The students still scored at significantly higher levels on exams than the students who worked systematically through every method (Boaler, 2016).

Mathematics educators have been highlighting the importance of big ideas for many years. In 1999, Deborah Schifter and her colleagues wrote an important chapter called "Teaching to the Big Ideas" and Randall Charles set out a helpful list of big ideas for K-8 mathematics in 2005. (Links to both of these resources are provided in the appendix.) Jere Confrey and her team at North Carolina State University have also set out K-8 learning trajectories, that are very compatible with big ideas and show the links across grade levels really nicely, (https://turnonccmath.net/).

In our work with teachers highlighting the importance of teaching to big ideas and the connections between them teachers frequently ask us – what are the big ideas? Many teachers were themselves taught mathematics as a list of content, missing understanding of the big ideas that make mathematics a cohesive whole. It is ideal for teachers to sit down and work out big unifying ideas together, but few teachers have the time to do this and many lack the colleagues that would make these conversations possible. When teachers work on identifying and discussing big ideas, they become attuned to the mathematics that is most important and that they may see in tasks, they also develop a greater appreciation of the connections that run between tasks and ideas.

As the three of us have been writing curriculum materials for K-8 teachers we also worked out a set of big ideas for each grade and a set of engaging activities to go with each, and this paper shares our big ideas in the hope they are helpful for teacher discussions. The big ideas we developed are not intended to be definitive and a different group could well have formed a different set of ideas, but we hope they will be useful for others to read and think about. The 4th grade book just released and the 5th grade will release in Feb 2018. (For details on the timing of other grades and how to buy the books see the end of this paper.)

Schedule for K-8 books

Grade 4: available now
Grade 5: available Feb 20 2018
Grade 3: August 2018
Grades 6-8: 2019
Grades K-2: 2020

Mindset Mathematics K-8 Series

For each big idea, our new curriculum materials have three activities, one that engages the students visually, one that engages them through an investigation that they can extend to any level and one through a play activity, allowing the ideas to go deeper. The activities we have developed are intended to supplement and enrich existing curriculum in schools.


Visualize


Play


Investigate

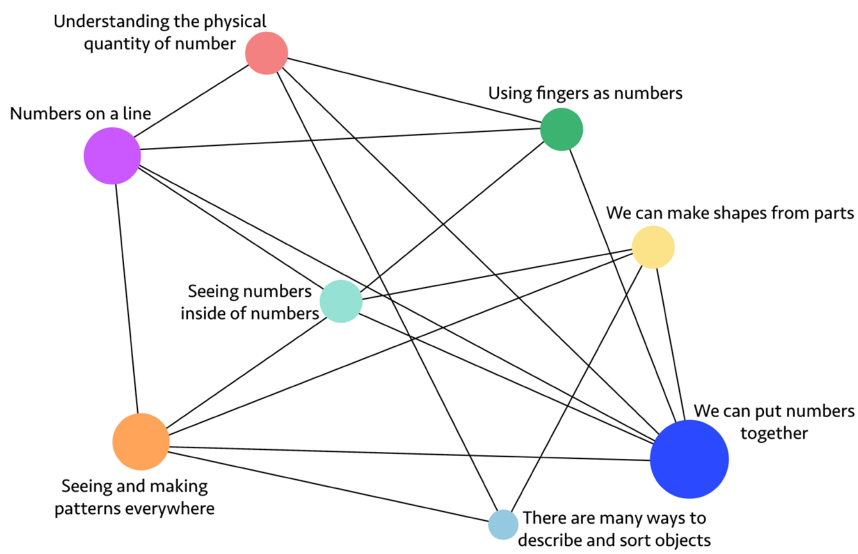
Framing the big ideas in each grade was a critical first step for us in writing curriculum. We couldn't begin to think about activities, puzzles, or investigations until we knew how to anchor the lessons into a coherent set of mathematical ideas. We began by looking at the Common Core State Standards and considered what connections existed within and across the standards that built on ideas from a previous grade and were pivotal to ideas in later grades. We also thought carefully about the ideas that get little attention in standards and curriculum, but that are powerful for mathematical thinkers, and we included those ideas in our materials.

We tested our big ideas by creating the networks that follow, knowing that if an idea is truly important, it will be connected to other ideas in the grade level. These connections give mathematics coherence which supports all students in making sense, as students draw on what they know about one big idea to learn about another. There are several ways you might use these maps of big ideas. You might get together with colleagues and discuss how you understand the meaning of each big idea and how it connects to what you teach. What opportunities do or could students have to explore these big ideas? How might you make these big ideas prominent in your teaching? You might also explore the connections within each network, and ask, how is this big idea connected to the others? How would you describe that connection? How could you give students the opportunity to connect these ideas? Finally, you might explore how the big ideas in one grade level are connected to those in previous or later grades. As we wrote these big ideas, it was clear to us that there are some even bigger ideas that pervade all of mathematics, such as looking for and naming patterns, examining and naming relationships, composing and decomposing with numbers and shapes, and so forth. Looking across grade levels is a useful way to begin to see the full mathematical terrain.

It is important to note that the big ideas we set out in this paper are living, and we anticipate that they will evolve with our thinking, writing, and conversations with teachers. For each network of big ideas, we've written a brief caption describing what we see as being at the heart of each grade level's mathematical work. We have also highlighted some ideas that might surprise you, or connections we want to draw your attention to. You will, no doubt, see many other points of interest in the big ideas, and we encourage you to look, think, talk, question, and explore.

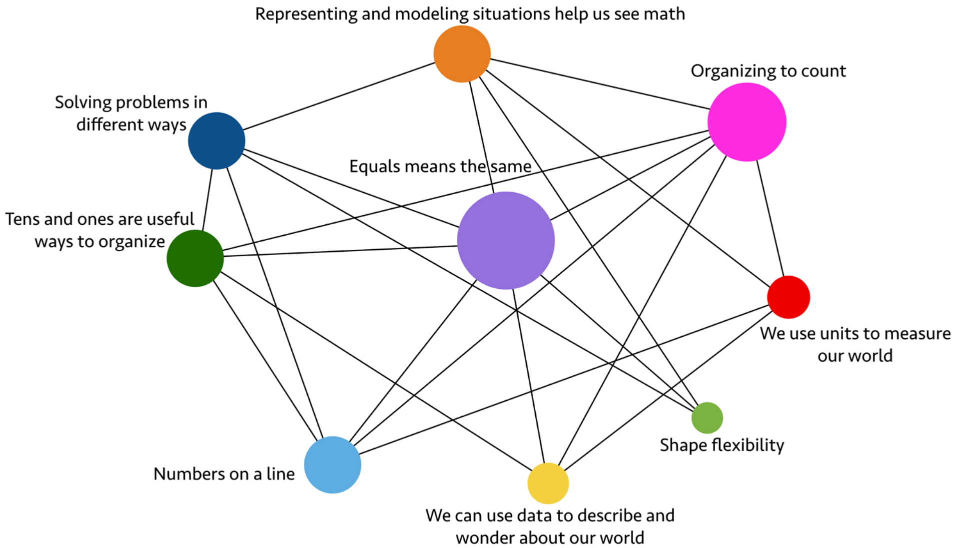
Big Ideas from Mindset Mathematics K-8 books

Kindergarten Big Ideas



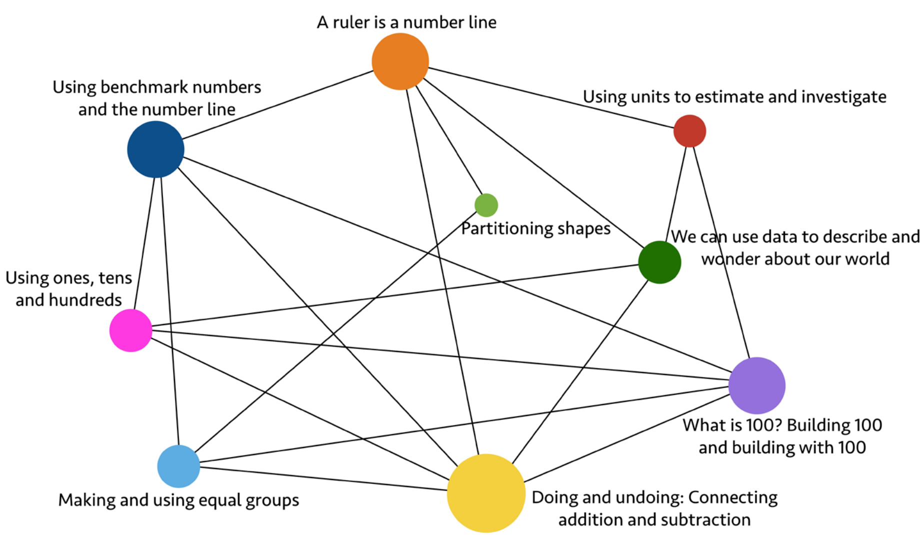
In kindergarten, students are working on figuring out what numbers mean – how numbers connect to fingers, objects, movement, and each other. Of particular importance is how numbers (and the objects they represent) and shapes can be put together and taken apart to create something new, but related. These are powerful early steps in encouraging students to look for and name mathematical connections. To learn more about why fingers are so critical to mathematics see Boaler & Chen (2016). https://www.theatlantic.com/education/archive/2016/04/why-kids-should-use-their-fingers-in-math-class/478053/

First Grade Big Ideas



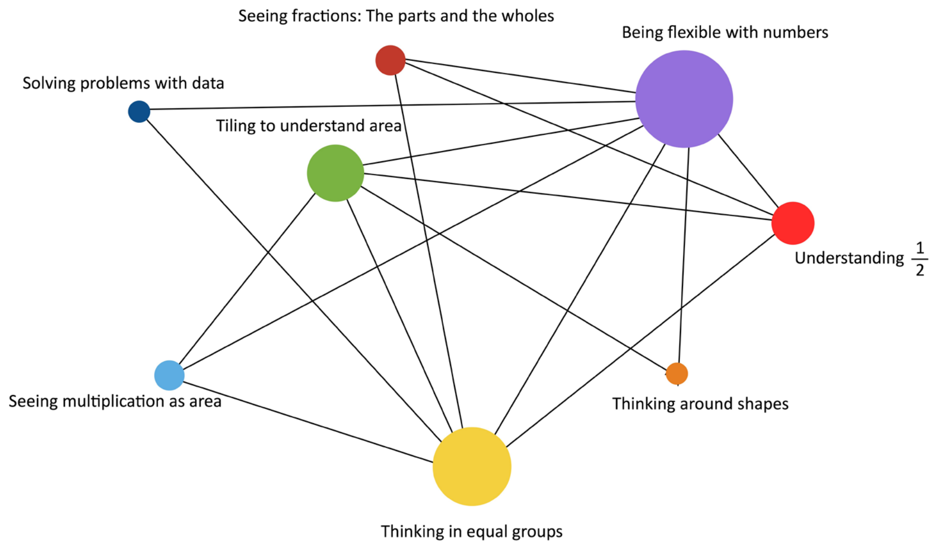
Organizing and seeing equivalence are ideas which pervade first grade. Students need to develop ways to organize for counting and comparing, and ultimately make meaning out of our place value system. Equivalence means learning to assess what makes things different and the same. For instance, 4 + 1 and 5 are equivalence, even though they look different, and students may develop a dozen strategies for adding 4 and 1 to arrive at 5. Those strategies are different, but related and equivalent in the result they produce. Grappling with equivalence and organization is important work in first grade.

Second Grade Big Ideas



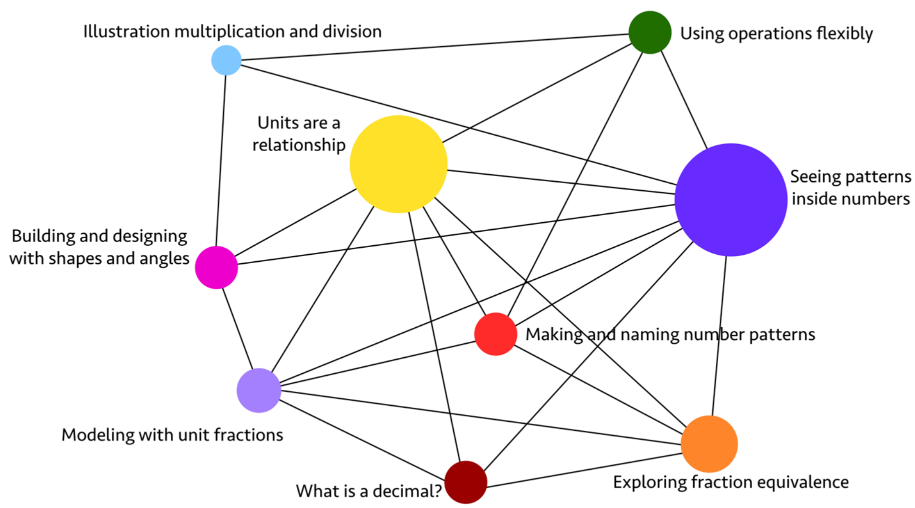
Second graders think deeply about making benchmark numbers familiar, so they can use them as tools to compose, decompose, and compare numbers. Thinking with pieces, like ones, tens, and hundreds, and negotiating how to use them as groups and as positions on the number line to solve problems is central to this grade. Students need to continually anchor their thinking about number to all the real-world places where numbers are used to describe and wonder, including estimating lengths and quantities and thinking with data.

Third Grade Big Ideas



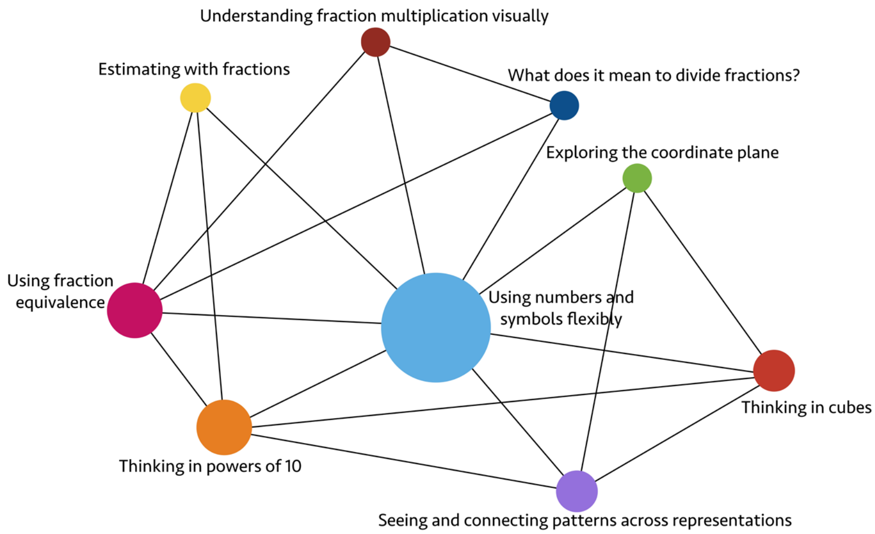
Some really big ideas take center stage in third grade. The work students have done to think with groups in second grade now takes the form of thinking about equal groups - and rows and columns – in multiplication. Being flexible with numbers is our answer to fluency; rather than a focus on being fast with computation, we position being flexible as a big idea, in which we use connections between numbers and the patterns we've noticed to support students in constructing a flexible internal framework for number relations that they can draw on when working with any mathematics. Third grade is also the time when fractional thinking begins to become robust, and this begins with a deep and flexible understanding of ½ that students can build on to understand and visualize other unit fractions. To learn more about why timed tests should be replaced with number sense activities see Boaler, WIliams & Confey, (2015) https://www.youcubed.org/evidence/fluency-without-fear/

Fourth Grade Big Ideas



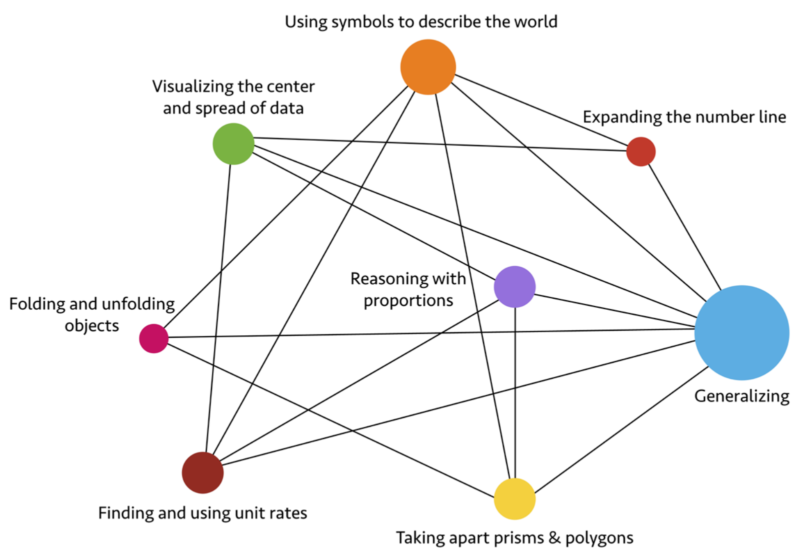
Patterning and examining relationships are at the heart of fourth grade. Students begin to think about how to identify and express patterns, both visually and numerically, and build foundations for proportional reasoning when thinking about the connections between units. Students look within fractions and decimals for the relationships represented there – relationships between numerator and denominator, fraction and decimal, and decimal and place value. Fourth graders use relationships to connect multiplication and division and think flexibly across all operations.

Fifth Grade Big Ideas



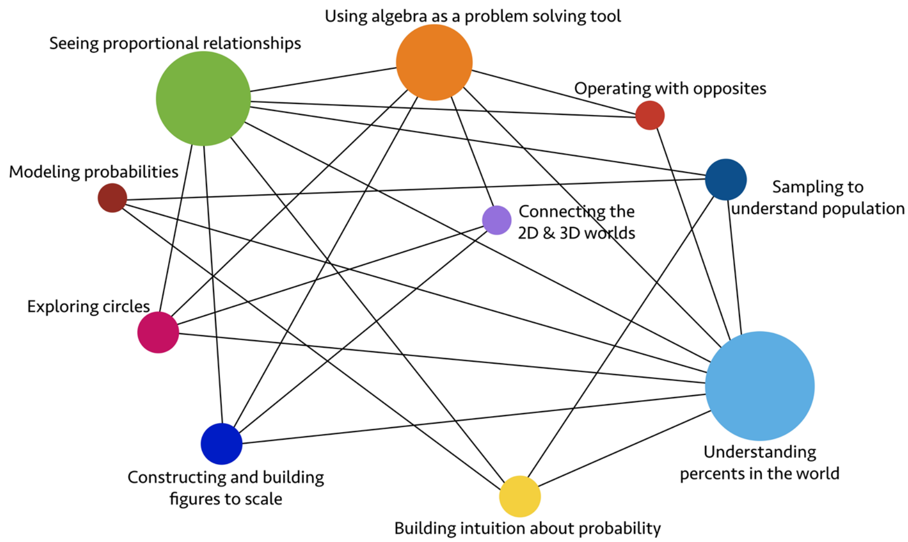
In fifth grade, students are working deeply on notions of equivalence and flexibility, both relating to operations and fractions, in particular. Fraction relationships and using relationships in the world to make meaning out of multiplication, division, and estimation require a great deal of exploration. We've added estimating with fractions because thinking about portions is a useful and underdeveloped idea that gives fractions meaning and utility. Moving to the fore in fifth grade are ideas about patterns and relationships in two- and three-dimensional space. Students begin to use the coordinate plane to represent and question relationships, and they begin to think about how to count and represent volume by anchoring themselves to cubic units.

Sixth Grade Big Ideas



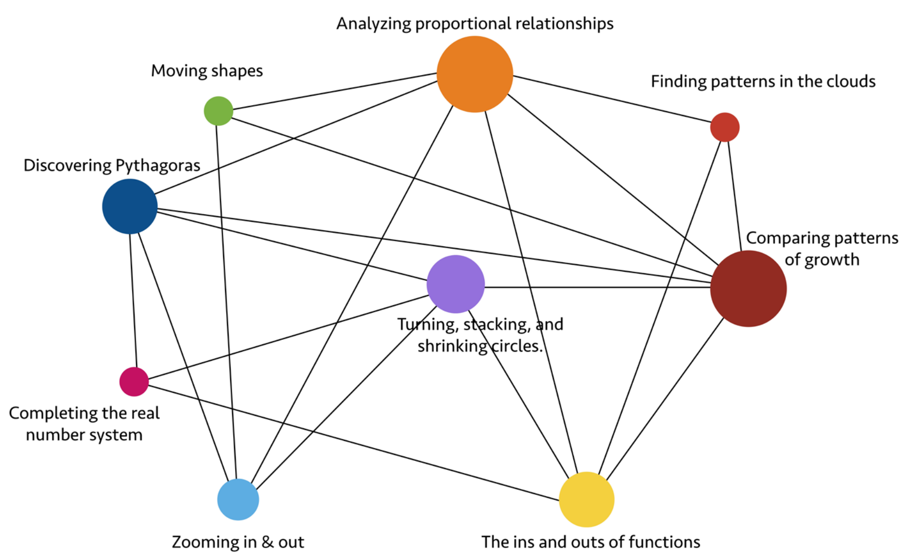
Proportional reasoning, unit rates, and generalizing relationships are central to sixth grade. This represents a major shift for students and is worthy of deep, sustained attention. Students build new ways to represent the world symbolically, on the number line, and through data that add nuance – both messiness and order – to the mathematical terrain. Sixth graders develop new ways to compose and decompose with two- and three-dimensional shapes, thinking about volume and area as additive and using nets to explore the surfaces that create solids.

Seventh Grade Big Ideas



Seventh grade includes a large new space devoted to percents and making sense of probability, each of which draw on students' proportional reasoning. Students connect proportional reasoning to the two- and three-dimensional world through the construction of scale figure. Algebra is used as a tool for solving problems, not just as a mathematical space of its own, and students can connect algebra back to the work they do with proportion, geometry, and the new world of integers.

Eighth Grade Big Ideas



Proportional relationships continue to be a hub of mathematical thought in eighth grade, serving as a tool for thinking about patterns of growth, functions, and geometric transformations. Functions are an important addition to the algebraic space in eighth grade. We see Pythagoras as a critical entry point for exploring the real number system, patterns of growth, and circles, which we have called Turning, Stacking, and Shrinking Circles. One big idea that challenges students notions of clean, linear relationships, after all the work on functions and proportions, is an idea we've called Finding patterns in the clouds. Data in the real world is rarely neat and lock-step; this is an important moment to develop a lens for looking at scatterplots and genuinely asking what relationships can be found in the clouds?

Conclusion

Our hope in providing these big ideas for K-8 is that they will encourage thinking about the threads that weave across mathematics and that are pivotal in students' mathematics learning. Many teachers, as well as students, have been given the faulty idea that mathematics is just a long list of methods and disconnected rules. We hope that our ideas will initiate rich conversations between teachers about the big ideas and the connections that relate them to each other. If you don't have colleagues to discuss the ideas with, (or even if you do) our youcubed Facebook group (https://www.facebook.com/groups/youcubed/) is a lovely space for collegial discussions. When teachers are thinking about big ideas they become more attuned to the most important tasks to choose for students and the connections to highlight in conversations and through teaching. Looking at the big ideas in different grades will also help teachers see what students have been learning and will be learning. We are now in the 21st century, student do not need to be trained to be calculators - we have technology for this - but they do need to experience mathematics as a beautiful and connected subject of enduring big ideas. Students who learn through big ideas and connections enjoy mathematics more, understand mathematics more deeply and are better prepared to tackle the big complex problems and discoveries that they will meet in their lives.

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Appendix/ Other Resources:

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http://jaymctighe.com/wordpress/wp-content/uploads/2011/04/MATH-Big-Ideas_NCSM_Spr05v73p9-24.pdf

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